Complexity and Random Polynomials

Diego Armentano

PEDECIBA, Universidad de la República
Université Paul Sabatier

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Advisors:
Jean-Pierre Dedieu (Université Paul Sabatier)
Michael Shub (PEDECIBA – CONICET, IMAS, Universidad de Buenos Aires)
Mario Wschebor (Universidad de la República)

Jury Members:
Jean-Marc Azaïs (Université Paul Sabatier)
Felipe Cucker (City University of Hong Kong)
Gregorio Malajovich (Universidade Federal do Rio de Janeiro)
Michael Shub (PEDECIBA – CONICET, IMAS, Universidad de Buenos Aires)
Jean-Claude Yakoubsohn (Université Paul Sabatier)

1. Rapporteur: *Felipe Cucker* (City University of Hong Kong).

2. Rapporteur: *Gregorio Malajovich* (Universidade Federal do Rio de Janeiro).

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Signatures of PhD Committee

Abstract

In this dissertation we analyze two different approaches to the problem of solving system of polynomial equations.

In the first part of this thesis we analyze the complexity of certain algorithms for solving system of equations, namely, *homotopic methods* or *path-following methods*. Special attention is given to the eigenvalue problem, introducing a projective framework to analyze this problem. The main result is to bound the complexity of path-following methods in terms of the length of the path in the condition metric, proving the existence of short paths in the condition metric. We also address the problem of the complexity of Bézout's theorem, reconsidering Smale's algorithm in the light of work done in the intervening years. At the end of this first part we define a new condition number adapted to directionally uniform perturbations in a general framework of maps between Riemannian manifolds, relating it with the classical condition number in many interesting examples.

In the second part of this dissertation we center our attention on the set of solutions of system of equations where the coefficients are taken at random with some probability distribution. We start giving an outline on Rice formulas for random fields. We review some recent results concerning the expected number of real roots of random systems of polynomial equations. We also recall and give new proofs of some known results about the undetermined case, that is, when the random system of equations has less equations than unknowns. We also study complex random systems of polynomial equations. We introduce the technics of Rice formulas in the realm of complex random

fields. In particular, we give a probabilistic approach of Bézout's theorem using Rice formulas. At the end of this second part we deal with the following question: How are the roots of complex random polynomials distributed?. We prove that points in the sphere associated with roots of random polynomials via the stereographic projection, are surprisingly well-suited with respect to the minimal logarithmic energy on the sphere. That is, roots of random polynomials provide a fairly good approximation to Elliptic Fekete points.

Résumé

Dans ce travail, nous étudions deux approches différentes pour de résoudre un système d'équations polynomiales.

Dans une première partie, nous analysons la complexité de certains algorithmes de résolution de systèmes d'équations, plus précisément des "méthodes d'homotopie" appelées aussi "méthodes de suivi de chemins".
Nous analysons spécialement le problème de la valeur propre, en le traitant dans un contexte projectif. Le résultat principal donne une borne à la complexité des méthodes de suivi de chemins en fonction de la longueur des chemins en la métrique du conditionnement, tout en prouvant l'existence de chemins courts dans cette métrique.
Nous traitons aussi le problème de la complexité du Théorème de Bézout, en re-comprenant l'algorithme de Smale à la lumière des années de travail qui ont suivi. À la fin de cette première partie, nous définissons une nouvelle notion de conditionnement, qui s'adapte à des perturbations uniformément directionnelles, dans un contexte général d'applications entre variétés de Riemann et nous montrons, sur plusieurs exemples intéressants, comment il est relié au conditionnement classique.

Dans une deuxième partie, nous étudions l'ensemble de solutions des systèmes d'équations dont les coefficients sont aléatoires. Nous commençons par donner une idée des formules de Rice pour des champs aléatoires réels et nous rappelons quelques résultats concernant le nombre moyen de racines réelles de systèmes d'équations polynomiales

aléatoires. Nous rappelons aussi quelques résultats connus sur le cas sous-déterminé (c'est à dire le cas où le système d'équations aléatoires a moins d'équations que de variables), en présentant quelques preuves nouvelles.
Nous étudions aussi des systèmes d'équations polynomiales aléatoires complexes, en introduisant des techniques de formules de Rice dans la théorie des champs aléatoires complexes. En particulier, nous donnons une approche probabiliste au Théorème de Bézout, en utilisant des formules de Rice. À la fin de cette deuxième partie, nous traitons la question suivante: comment sont distribuées les racines des polynômes complexes aléatoires? Nous prouvons que certains points de la sphère associés à des racines de polynômes aléatoires à travers la projection stéréographique sont étonnamment bien placés par rapport à l'énergie logarithmique minimale de la sphère. C'est à dire, les racines de polynômes aléatoires donnent une bonne approximation des points de Fekete elliptiques.

Resumen

En esta disertación analizamos dos enfoques diferentes para el problema de resolver sistemas de ecuaciones polinomiales.

En la primer parte de esta memoria analizamos la complejidad de ciertos algoritmos para resolver sistemas de ecuaciones, a saber, *métodos homotópicos* o *métodos de seguimiento de caminos*. Ponemos especial atención al problema de valores propios, introduciendo un marco proyectivo para analizar este problema. El resultado principal es acotar la complejidad de caminos de homotopía en términos de la longitud del camino en la métrica de condición. También estudiaremos el problema de la complejidad del teorema de Bézout, reconsiderando el algoritmo de Smale en la luz del trabajo hecho en los últimos años. Al final de esta primera parte definimos un nuevo número de condición adaptado a perturbaciones con direcciones uniformes en un contexto general entre variedades Riemannianas, relacionándolo con los números de condición clásicos en varios ejemplos interesantes.

En la segunda parte de esta memoria nos concentramos en las soluciones de sistemas de ecuaciones cuando los coeficientes de estos son tomados al azar con cierta distribución de probabilidad. Empezaremos dando una breve reseña sobre la fórmula de Rice para campos aleatorios. Repasaremos algunos resultados recientes relacionados al número esperado de raíces reales de un sistema de ecuaciones polinomiales. También repasaremos, dando nuevas pruebas, algunos resultados conocidos relacionados al caso indeterminado, es decir, cuando el sistema de ecuaciones aleatorias tiene más variables que ecuaciones. También estudiaremos sistemas polinomiales aleatorios complejos. Introduciremos las técnicas de Rice en la teoría de campos aleatorios

complejos. En particular, daremos un enfoque probabilísta al teorema de Bézout usando las fórmulas de Rice. En el final de esta segunda parte consideramos el siguiente problema: ¿cómo están distribuidas las raíces de polinomios complejos aleatorios? Probaremos que puntos en la esfera asociados a raíces de polinomios complejos aleatorios están sorprendentemente bien distribuídos con respecto al mínimo de la energía logarítmica sobre la esfera. Esto es, raíces de polinomios aleatorios brindan una muy buena aproximación de los puntos de Fekete elípticos.

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Chapter 0

Introduction

The problem of solving systems of polynomial equations is a classical subject with a long history. This problem has decisively influenced in the discovery of complex numbers and group theory, and was one of the main motivations in the development of *Algebraic Geometry* and *Algebra*.

This thesis is intimately related with the problem of solving systems of polynomial equations. Precisely, we will pursue two different aspects of this problem. In the first part of this dissertation we analyze the complexity of certain algorithms for solving system of equations, namely, *homotopic methods* or *path-following methods*. In the second part of this dissertation we center our attention on the set of solutions of system of equations where the coefficients are taken at random with some probability distribution.

In the following two sections we outline these two approaches and we explicit the main contributions of this dissertation.

0.1 Complexity of Algorithms and Numerical Analysis

Since Abel and Galois the unsolvability of polynomials of degree bigger than four in terms of radicals has been known. Thereby, iterative methods play a leading role in the study of this problem. Regarding this matter, approximating solutions of systems of equations is one of the main activities in *Numerical Analysis*, and is one of the cornerstones of the foundation of the *Complexity of Algorihtms*.

A good measure of the complexity of an algorithm is the number of arithmetic operations required to pass from the input to the output. The problem of studying the complexity of algorithms has a long tradition in computer science, where the discrete mathematics of Turing machines are the underlying mathematics. But it was not until the early 80's, that Steve Smale made an important contribution in the theory, with his pionering paper Fundamental Theorem of Algebra [Smale, 1981], bringing the continuous mathematics of classical analysis and geometry to this field.

On the other hand, until that moment, the tradition in *Numerical Analysis* to study iterative methods was divided in a 2-part scheme: proof of convergence, and asymptotic speed of convergence.

In his 1981 paper, Smale proposed a probabilistic analysis of complexity for a certain variant of Newton's method. Quoting Smale [1981]:

> " ...the Newton type methods fail in principle for certain degenerate cases. And near the degenerate cases, these methods are very slow. This motivates a statistical theory of cost, i.e. one which applies to most problems in the sense of a probabilistic measure on the set of problems (or data). There seems to be a trade off between speed and certainty, and a question is how to make that precise."

Smale [1997] suggested a systematic way to analyze the complexity of an algorithm where the *condition number* plays a prominent role. Roughly, the condition number x is a measure of how close to the space of degenerate inputs x is.

The 2-part scheme suggested by Smale [1997], to analyze the complexity of an algorithm, is the following:

1. Given an input x, bound the number of arithmetic opeartations $K(x)$ by

$$K(x) \leq (\log \mu(x) + \mathrm{size}(x))^c \, ,$$

where c is a universal constant, size(x) is the size of the input x, and μ is the condition number.

2. Estimate the probability distribution of μ, where the tail takes the form

$$P(\mu(x) > \varepsilon^{-1}) \leq \varepsilon^{c},$$

for some probability measure on the space of inputs.

Key questions such as: *What are the most efficient algorithms?* or *Which algorithms have polynomial average complexity?* can be addressed, building in this way the foundations of complexity of numerical analysis.

During the last three decades, an enormous amount of work has been done on this scheme for complexity of polynomial system solving. Let us mention a few changes.

In their seminal paper Shub & Smale [1993a] relate, in the context of polynomial system solving, the complexity K to three ingredients: the degree of the considered system, the length of the path $\Gamma(t)$, and the condition number of the path. Precisely, they obtain the complexity

$$K \leq CD^{3/2}\ell(\Gamma)\mu(\Gamma)^2,$$

where C is a universal constant, D is the degree of the system, $\ell(\Gamma)$ is the length of Γ in the associated Riemannian structure, and $\mu(\Gamma) = \sup_{a\leq t\leq b} \mu\left(\Gamma(t)\right)$.

In Shub [2009] the complexity K of path-following methods for the polynomial system solving problem is analyzed in terms of the condition length of the path.

It is in this spirit that the first part of this dissertation is developed.

0.1.1 Preliminaries

Before the statements of the main contributions of this thesis we introduce the basic definitions associated to a computational problem.

The Varieties $\mathcal{V}$, Σ' and Σ

Let $\mathcal{X}$ and $\mathcal{Y}$ be the spaces of *inputs* and *outputs* associated respectively to some computational problem. In this thesis, the spaces $\mathcal{X}$ and $\mathcal{Y}$ are linear or differential manifolds.

Suppose that $\mathcal{X}$ and $\mathcal{Y}$ are real (or complex) finite dimensional manifolds such that $\dim \mathcal{X} \geq \dim \mathcal{Y}$.

The *solution variety* $\mathcal{V} \subset \mathcal{X} \times \mathcal{Y}$ is the subset of pairs $(x, y) \in \mathcal{X} \times \mathcal{Y}$ such that y is an output corresponding to the input x.

Let $\pi_1 : \mathcal{V} \to \mathcal{X}$ and $\pi_2 : \mathcal{V} \to \mathcal{Y}$ be the restrictions to the solution variety $\mathcal{V}$ of the canonical projections (see the diagram).

$$\begin{tikzcd} & \mathcal{V} \subset \mathcal{X} \times \mathcal{Y} \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ \mathcal{X} & & \mathcal{Y} \end{tikzcd}$$

Note that algorithms attempt to "invert" the projection map π_1, hence, the subset of critical points of the projection π_1 plays a central role in complexity of algorithms.

Let $D\pi_1(x, y) : T_{(x,y)}\mathcal{V} \to T_x\mathcal{X}$ be the derivative of π_1 and let Σ' be the subset of critical points of π_1, that is,

$$\Sigma' := \{(x, y) \in \mathcal{V} : \ \mathrm{rank} D\pi_1(x, y) < \dim \mathcal{X}\}.$$

Σ' is called the *ill-posed variety* or *critical variety.*

Let

$$\Sigma := \pi_1(\Sigma') \subset \mathcal{X},$$

be the set of *ill-posed inputs* or *discriminant variety.*

In order to have local uniqueness of the "inverse" of π_1, a reasonable hypothesis is to assume that the $\dim \mathcal{V} = \dim \mathcal{X}$. When this is the case, according to the implicit function theorem, for each $(x, y) \in \mathcal{V} \backslash \Sigma'$ there is a differentiable function locally defined between some neighborhoods U_x and U_y of $x \in \mathcal{X}$ and $y \in \mathcal{Y}$ respectively, namely, the *solution map*

$$\mathscr{S}(x, y) := \pi_2 \circ \pi_1^{-1}|_{U_x} : U_x \to U_y.$$

Its derivative

$$D\mathscr{S}(x, y) : T_x\mathcal{X} \to T_y\mathcal{Y},$$

is the *condition operator*.

The Condition Number

Assume that $\mathcal{X}$ and $\mathcal{Y}$ are Riemannian (or Hermitian) manifolds. Let $\langle\cdot,\cdot\rangle_x$ and $\langle\cdot,\cdot\rangle_y$ be the Riemannian (or Hermitian) inner product in the tangent spaces $T_x\mathcal{X}$ and $T_y\mathcal{Y}$ at x and y respectively.

The *condition number* at $(x, y) \in \mathcal{V} \setminus \Sigma'$ is defined as:

$$\mu(x, y) := \max_{\substack{\dot{x} \in T_x\mathcal{X} \\ \|\dot{x}\|_x^2 = 1}} \|D\mathscr{S}(x, y)\dot{x}\|_y.$$

This number is an upper-bound -to first-order approximation- of the *worst-case* sensitivity of the output error with respect to small perturbations of the input. There is an extensive literature about the role of the condition number in the accuracy of algorithms, see for example Higham [1996] and references therein.

Remark: This general framework of maps between Riemannian manifolds was motivated by Shub & Smale [1996] and Dedieu [1996]. This framework for a *computational problem* differs from the usual one, where the problem being solved can be described by a univalent function $\mathscr{S}$. In the given context, we allow multivalued functions, that is, we allow inputs with different outputs. In this way, one can define the condition number for the input $x \in \mathcal{X}$ as a certain functional defined over $(\mu(x, y))_{\{y \in \pi_2(\pi_1^{-1}(x))\}}$. When the function $\mathscr{S}$ is univalent the condition number $\mu(x) := \mu(x, y)$ coincides with the classical condition number (see Higham [1996], pag. 8).

In this thesis we will restrict ourselves to a particular family of computational problems, namely, the problem of finding roots of systems of polynomial equations. Therefore, in this case, the space of inputs $\mathcal{X}$ is a certain subspace of system of polynomial equations over some field, and the space of outputs $\mathcal{Y}$ is associated to the set of "all" possible solutions. The solution variety $\mathcal{V} = \mathrm{ev}^{-1}(0)$, where ev is the evaluation map, i.e. $\mathrm{ev}(F, z) = F(z)$ for $F \in \mathcal{X}$ and $z \in \mathcal{Y}$.

In Malajovich [2011] one can find an extension to the problem of finding roots of analytic equations.

Path-Following Methods

Let $F \in \mathcal{X}$ be a system of equations one wishes to solve. Roughly, *path-following methods* or *homotopy methods* consists in considering a new system F_0, with a prescribed solution $z_0 \in \mathcal{Y}$, and then attempting to approximate the (π_1) lifted path $(F_t, z_t) \in \mathcal{V}$, $0 \leq t \leq 1$, of some path $F_t \in \mathcal{X}$ joining F_0 with $F = F_1$. If this procedure succeeds, then z_1 is a solution of our problem.

The lift of the path F_t, $0 \leq t \leq 1$, by the projection π_1, exists provided that $F_t \in \mathcal{X} \setminus \Sigma$ for all $0 \leq t \leq 1$.

The algorithmic way to do this procedure is to construct a finite number of pairs

$$(F_{t_k}, z'_{t_k}), \qquad 0 = t_0 \leq t_k \leq t_K = 1,$$

such that z'_{t_k} is an approximation of z_{t_k}.

A possible scheme to find the approximations z'_{t_k} is to consider a *predictor-corrector algorithm* (cf. Allgower & Georg [1990]).

In this thesis we will be mainly concern with the following approximation:

$$z'_{t_{k+1}} := N_{F_{t_{k+1}}}(z_{t_k}),$$

where N_F denotes the *Newton map* associated to the system F.

For a detailed account in path-following methods see Allgower & Georg [1990].

Canonical Hermitian Structures

Given a finite dimensional vector space V over $\mathbb{K}$ with the Hermitian inner product $\langle \cdot, \cdot \rangle$ and $0 \neq v \in V$, we let

$$v^{\perp} = \{w \in V : \langle w, v \rangle = 0\}.$$

The vector space $v^{\perp}$ is a model for the tangent space $T_v\mathbb{P}(V)$, of the projective space $\mathbb{P}(V)$ at the equivalence class of v (which we denote by v).

In this way we can define an Hermitian structure over $\mathbb{P}(V)$ in the following way: for $v \in V$,

$$\langle w, w' \rangle_v := \frac{\langle w, w' \rangle}{\|v\|^2},$$

for all w, $w' \in v^{\perp}$.

The space $\mathbb{K}^{\ell}$ is equipped with the canonical Hermitian inner product $\langle \cdot, \cdot \rangle$, namely

$$\langle x, y \rangle = \sum_{k=0}^{\ell} x_k \, \overline{y_k}.$$

The space $\mathbb{K}^{n \times n}$ is equipped with the Frobenius Hermitian inner product

$$\langle A, B \rangle_F := \text{trace }(B^* A),$$

where B^* denotes the adjoint of B.

0.1.2 Main Contributions

In this section we introduce the main contributions given in this thesis associated to the complexity of algorithms in numerical analysis.

0.1.2.1 Complexity of The Eigenvalue Problem

The eigenvalue problem is the problem to solve, for a fixed matrix $A \in \mathbb{K}^{n \times n}$, the following system of polynomial equations:

$$(\lambda I_n - A)v = 0, \quad v \neq 0,$$

where $v \in \mathbb{K}^n$, $\lambda \in \mathbb{K}$. Here $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

Classical algorithms for solving the eigenvalue problem may be divided into two classes: QR methods and Krylov subspace methods.

Even these methods have a long history, surprisingly, the complexity of the eigenvalue problem is still an open problem.

In this thesis we will study path-following methods for the eigenvalue problem.

In the context of polynomial system solving, the eigenvalue problem may be considered as a quadratic system of equations. However, Shub & Smale [1993a] and Shub & Smale [1996] do not apply since the eigenvalue problem as a quadratic system belongs to the subset of ill-posed problems of generic quadratic systems. (See Li [1997]). Therefore, in order to analyze the complexity of the eigenvalue problem, a different framework is required. Here we consider the eigenvalue problem as a bilinear problem.

In *Chapter 1*, following Armentano [2011a], we introduce a projective framework to analyze this problem. We define a condition number and a Newton's map appropriate for this context, proving a version of the γ-Theorem and a condition number theorem for this context. The main result in *Chapter 1* is to bound the complexity of path-following methods in terms of the length of the path in the condition metric.

Let us outline some results.

Since the system of equations $(\lambda I_n - A)v = 0$ is homogeneous in $v \in \mathbb{K}^n$ and also in $(A, \lambda) \in \mathbb{K}^{n\times n} \times \mathbb{K}$, we define the solution variety as

$$\mathcal{V} =: \left\{(A, \lambda, v) \in \mathbb{P}\left(\mathbb{K}^{n\times n} \times \mathbb{K}\right) \times \mathbb{P}\left(\mathbb{K}^n\right) : \ (\lambda I_n - A)v = 0\right\}.$$

The solution variety $\mathcal{V}$ is bi-projective algebraic subvariety of $\mathbb{P}\big(\mathbb{K}^{n\times n} \times \mathbb{K}\big) \times \mathbb{P}(\mathbb{K}^n)$. Moreover, $\mathcal{V}$ is also a smooth manifold and its dimension is equal to the dimension of $\mathbb{P}(\mathbb{K}^{n\times n})$.

Note that the solution variety differs from the general setting defined in the *Preliminaries*. However, as we will see in *Chapter 1* we can define a natural projection $\pi : \mathcal{V} \to \mathbb{P}(\mathbb{K}^{n\times n})$, given by $\pi(A, \lambda, v) = A$. In this way, we may consider the space $\mathbb{P}(\mathbb{K}^{n\times n})$ as the input space, and hence, we may proceed as in the *Preliminaries* section.

Let $\mathcal{W} \subset \mathcal{V}$ be the set of *well-posed* problems. It is not difficult to prove that $\mathcal{W}$ is the set of triples $(A, \lambda, v) \in \mathcal{V}$ such that λ is a simple eigenvalue. In that case, the operator $\Pi_{v^\perp}(\lambda I_n - A)|_{v^\perp}$ is invertible, where $\Pi_{v^\perp}$ denotes the orthogonal projection of $\mathbb{K}^n$ onto $v^\perp$.

As in the *Preliminaries* section, when (A, λ, v) belongs to $\mathcal{W}$, we can define the solution map $\mathscr{S} = \pi^{-1}|_{\mathcal{U}_A} : \mathcal{U}_A \to \mathcal{V}$ defined in some neighborhood $\mathcal{U}_A \subset \mathbb{P}(\mathbb{K}^{n\times n})$ of A such that $\pi^{-1}(A) = (A, \lambda, v)$. It associates to any matrix $B \in \mathcal{U}_A$ the eigentriple (B, λ_B, v_B) close to (A, λ, v). Note that one can decompose $\mathscr{S}$ in two solutions maps, namely, the solution map of the eigenvalue given by $\mathscr{S}_\lambda(B) = (B, \lambda_B)$, and the solution map of the eigenvector given by $\mathscr{S}_v(B) = v_B$.

The space $\mathbb{P}\big(\mathbb{K}^{n\times n} \times \mathbb{K}\big) \times \mathbb{P}(\mathbb{K}^n)$ inherits the Hermitian product structure $\|(\dot{A}, \dot{\lambda}, \dot{v})\|^2_{(A,\lambda,v)} = \|(\dot{A}, \dot{\lambda})\|^2_{(A,\lambda)} + \|\dot{v}\|^2_v$ for all $(\dot{A}, \dot{\lambda}, \dot{v}) \in (A, \lambda)^\perp \times v^\perp$.

Then we can define the condition numbers of the eigenvalue and eigenvector in the following way:

$$\mu_\lambda(A, \lambda, v) = \sup_{\substack{\dot{B} \in A^\perp \\ \|\dot{B}\|_F = \|A\|_F}} \|D\mathscr{S}_\lambda(A, \lambda, v)\dot{B}\|_{(A,\lambda)}$$

$$\mu_v(A, \lambda, v) = \sup_{\substack{\dot{B} \in A^\perp \\ \|\dot{B}\|_F = \|A\|_F}} \|D\mathscr{S}_v(A, \lambda, v)\dot{B}\|_v$$

Then, for $(A, \lambda, v) \in \mathcal{W}$ we obtain:

$$\mu_\lambda(A, \lambda, v) = \frac{1}{1 + \frac{|\lambda|^2}{\|A\|_F^2}} \cdot \left[1 + \frac{\|v\|^2 \cdot \|u\|^2}{|\langle v, u\rangle|^2}\right]^{1/2};$$

$$\mu_v(A, \lambda, v) = \|A\|_F \cdot \|\Pi_{v^\perp}(\lambda I_n - A)|_{v^\perp}{}^{-1}\|,$$

where $0 \neq u \in \mathbb{K}^n$ is a left eigenvector of A with associate eigenvalue λ, $\|\cdot\|_F$ and $\|\cdot\|$ are the Frobenius and operator norms in the space of matrices.

Let $(A, \lambda, v) \in \mathcal{W}$. If $(\lambda I_n - A)^* v = 0$, that is, if v is also a left eigenvector of A with eigenvalue λ, then,

$$\mu_\lambda(A, \lambda, v) = \frac{\sqrt{2}}{1 + \frac{|\lambda|^2}{\|A\|_F^2}}.$$

This is the case when A is normal, i.e. $A^*A = AA^*$. On the other hand, μ_v happens to be more interesting since, roughly speaking, it measures how close to λ others eigenvalues are.

In particular, when $A \in \mathbb{P}(\mathbb{K}^{n\times n})$ is a normal matrix, and $(A, \lambda, v) \in \mathcal{W}$ then

$$\mu_v(A, \lambda, v) = \frac{\|A\|_F}{\min_i |\lambda - \lambda_i|},$$

where the minimum is taken for λ_i an eigenvalue of A different from λ.

As we will see in *Chapter 1*, the condition number μ_λ is somehow controlled by μ_v. Thereby, we define the condition number of the eigenvalue problem at $(A, \lambda, v) \in \mathcal{W}$ as

$$\mu(A, \lambda, v) := \max\{1, \mu_v(A, \lambda, v)\}.$$

In *Chapter 1* we prove that, for $(A, \lambda, v) \in \mathcal{W}$, one get

$$\mu(A, \lambda, v) \leq \frac{1}{\sin(d_{\mathbb{P}^2}((A, \lambda, v), \Sigma'))}.$$

In the literature, these type of results relating the condition number to the distance to ill-posedness are known as the *Condition Number Theorems*.

From the point of view of complexity, these type of results are important to obtain probability estimates of the condition number. (See Smale [1981]).

When $\Gamma(t)$, $a \leq t \leq b$, is an absolutely continuous path in $\mathcal{W}$, we define its *condition-length* as

$$\ell_\mu(\Gamma) := \int_a^b \left\|\dot{\Gamma}(t)\right\|_{\Gamma(t)} \cdot \mu(\Gamma(t))\, dt,$$

where $\left\|\dot{\Gamma}(t)\right\|_{\Gamma(t)}$ is the norm of $\dot{\Gamma}(t)$ is the Riemannian structure on $\mathcal{V}$, inherited from $\mathbb{P}(\mathbb{K}^{n\times n} \times \mathbb{K}) \times \mathbb{P}(\mathbb{K}^n)$.

The main result in *Chapter 1* is:

There is a universal constant $C > 0$ such that for any absolutely continuous path Γ in $\mathcal{W}$, there exists a sequence which approximates

Γ, *and such that the complexity of the sequence is*

$$K \leq C\,\ell_\mu(\Gamma) + 1.$$

(One may choose $C = 120$*).*

This result motivates the study of geodesics in the condition length structure on $\mathcal{V}$ for the eigenvalue problem. This seems to be a very hard problem.

In *Chapter 2* we address the problem of the existence of short paths in the condition metric.

Let $\{e_1, \ldots, e_n\}$ be the canonical basis of $\mathbb{K}^n$, and let G be the rank one matrix $G := e_1 \cdot e_1^* \in \mathbb{K}^{n\times n}$. Let $\mathcal{W}_0$ be the set of problems $(A, \lambda, v) \in \mathcal{W}$ such that $\mu(A, \lambda, v) = 1$. Notice that $(G, 1, e_1) \in \mathcal{W}_0$. Then the main result of *Chapter 2* is the following:

For every problem $(A, \lambda, v) \in \mathcal{W}$ *there exist a path* Γ *in* $\mathcal{W}$ *joining* (A, λ, v) *with* $(G, 1, e_1)$*, and such that*

$$\ell_\mu(\Gamma) \quad \leq C\sqrt{n} \cdot (C' + \log(\mu(A, \lambda, v))),$$

for some universal constant C *and* C'*.*
(One may choose $C \leq \sqrt{6}$ *and* $C' \leq 10$*.)*

0.1.2.2 Complexity of Bezout's Theorem

In his 1981 Fundamental Theorem of Algebra paper Steve Smale initiated the complexity theory of finding a solution of polynomial equations of one complex variable by a variant of Newton's method.

Smale's algorithm may be given the following interpretation. For $z_0 \in \mathbb{C}$, consider $f_t = f - (1-t)f(z_0)$, for $0 \leq t \leq 1$. f_t is a polynomial of the same degree as f, z_0 is a zero of f_0 and $f_1 = f$. So, we analytically continue z_0 to z_t a zero of f_t. For $t = 1$ we arrive at a zero of f. Newton's method is then used to produce a discrete numerical approximation to the path (f_t, z_t).

Smale's result was not finite average cost. In the series of papers 1993a, 1993b, 1993c, 1996, Shub & Smale made some further changes and achieved enough results for Smale 17th problem to emerge a reasonable if challenging research goal. Let us recall the 17th problem from Smale [2000]:

Problem 17: Solving Polynomial Equations.

> *Can a zero of n-complex polynomial equations in n-unknowns be found approximately, on the average, in polynomial time with a uniform algorithm?*

In *Chapter 3*, following a joint work with Michael Shub (c.f. Armentano & Shub [2012]), we reconsider Smale's algorithm in the light of work done in the intervening years about this problem.

In the following lines we will give an outline of the main result.

Let $\mathcal{H}_{(d)} = \mathcal{H}_{d_1} \times \cdots \times \mathcal{H}_{d_n}$ where $\mathcal{H}_{d_i}$ is the vector space of homogeneous polynomials of degree d_i in $n+1$ complex variables.

On $\mathcal{H}_{d_i}$ we put a unitarily invariant Hermitian structure which we first encountered in the book Weyl [1939] and which is sometimes called Weyl, Bombieri-Weyl or Kostlan Hermitian structure depending on the applications considered.

For $\alpha = (\alpha_0, \ldots, \alpha_n) \in \mathbb{N}^{n+1}$, $\|\alpha\| = d_i$ the monomial $x^\alpha = x_0^{\alpha_0} \cdots x_n^{\alpha_n}$, the Weyl Hermitian structure makes $\langle x^\alpha, x^\beta \rangle = 0$, for $\alpha \neq \beta$ and

$$\langle x^\alpha, x^\alpha \rangle = \binom{d_i}{\alpha}^{-1} = \left(\frac{d_i!}{\alpha_0! \cdots \alpha_n!} \right)^{-1}.$$

On $\mathcal{H}_{(d)}$ we put the product structure

$$\langle f, g \rangle = \sum_{i=1}^{n} \langle f_i, g_i \rangle.$$

Given $\zeta \in \mathbb{P}(\mathbb{C}^{n+1})$ we define for $f \in \mathcal{H}_{(d)}$ the straight line segment $f_t \in \mathcal{H}_{(d)}$, $0 \leq t \leq 1$, by

$$f_t = f - (1-t)\Delta \left(\frac{\langle \cdot, \zeta \rangle^{d_i}}{\langle \zeta, \zeta \rangle^{d_i}} \right) f(\zeta),$$

where $\Delta(a_i)$ means the diagonal matrix whose i-th diagonal entry is a_i. So $f_0(\zeta) = 0$ and $f_1 = f$. Therefore we may apply homotopy methods to this line segment.

Note that if we restrict f to the affine chart $\zeta + \zeta^{\perp}$ then

$$f_t(z) = f(z) - (1-t)f(\zeta),$$

and if we take $\zeta = (1, 0, \dots, 0)$ and $n = 1$ we recover Smale's algorithm.

Let $f_t = f - (1-t)\Delta\left(\frac{\langle \cdot, \zeta\rangle^{d_i}}{\langle \zeta, \zeta\rangle^{d_i}}\right) f(\zeta)$, for $t \in [0,1]$, and ζ_t the homotopy continuation of ζ along the path f_t.

Suppose η is a non-degenerate zero of $f \in \mathcal{H}_{(d)}$. We define the basin of η, $B(f,\eta)$, as those $\zeta \in \mathbb{P}(\mathbb{C}^{n+1})$ such that the zero ζ of f_0 continues to η for the homotopy f_t.

The main result In *Chapter 3* is the following:

The average number of steps to follow the path $\{(f_t, \zeta_t) : t \in [0,1]\}$ *is bounded above by*

$$\text{(I)}\frac{CD^{3/2}\Gamma(n+1)2^{n-1}}{(2\pi)^N \pi^n} \int_{h \in \mathcal{H}_{(d)}} \Big[\sum_{\eta /\, h(\eta)=0} \frac{\mu^2(h,\eta)}{\|h\|^2}\Theta(h,\eta)\Big] e^{-\|h\|^2/2}\, dh,$$

where

$$\Theta(h,\eta) = \int_{\zeta \in B(h,\eta)} \frac{\left(\|h\|^2 - \|\Delta(\|\zeta\|^{-d_i})h(\zeta)\|^2\right)^{1/2}}{\|\Delta(\|\zeta\|^{-d_i})h(\zeta)\|^{2n-1}} \cdot \\ \cdot\, \Gamma(\|\Delta(\|\zeta\|^{-d_i})h(\zeta)\|^2/2, n) e^{\|\Delta(\|\zeta\|^{-d_i})h(\zeta)\|^2/2}\, d\zeta,$$

and $\Gamma(\alpha, n) = \int_{\alpha}^{+\infty} t^{n-1}e^{-t}\, dt$ *is the incomplete gamma function.*

This result may be helpful for Smale 17th problem and raises more problems than it solves.

(a) Is (I) finite for all or some n?

(b) Might (I) even be polynomial in N for some range of dimensions and degrees?

(c) What are the basins like? Even for $n = 1$ these are interesting questions.

The integral

$$\frac{1}{(2\pi)^N}\int_{h\in\mathcal{H}_{(d)}}\sum_{\eta/\,h(\eta)=0}\frac{\mu^2(h,\eta)}{\|h\|^2}\cdot e^{-\|h\|^2/2}\,dh\le\frac{e(n+1)}{2}\mathcal{D},$$

where $\mathcal{D}=d_1\cdots d_n$ is the Bézout number (see Bürgisser & Cucker [2011]). So the question is how does the factor $\Theta(h,\eta)$ affect the integral.

(d) Evaluate or estimate

$$\int_{\zeta\in\mathbb{P}(\mathbb{C}^{n+1})}\frac{1}{\|\Delta(\|\zeta\|^{-d_i})h(\zeta)\|^{2n-1}}\cdot e^{\frac{1}{2}\|\Delta(\|\zeta\|^{-d_i})h(\zeta)\|^2}\,d\zeta.$$

Note that

$$\|h\|_{L^p}=\left(\frac{1}{\mathrm{vol}(\mathbb{P}(\mathbb{C}^{n+1}))}\int_{\zeta\in\mathbb{P}(\mathbb{C}^{n+1})}\|\Delta(\|\zeta\|^{-d_i})h(\zeta)\|^p\,d\zeta\right)^{1/p},$$

for $p\ge 1$, is a different way to define a norm on h. For $p=2$ we get another unitarily invariant Hermitian structure on $\mathcal{H}_{(d)}$, which differs from the Bombieri-Weyl by

$$\|h\|_{L^2}^2=\sum_{i=1}^{n}\frac{d_i!n!}{(d_i+n)!}\|h_i\|^2,$$

(cf. [Dedieu, 2006, page 133])

If the integral in (d) can be controlled, if the integral on the $\mathcal{D}$ basins are reasonably balanced, the factor of $\mathcal{D}$ in (c) may cancel.

See *Chapter 3* for more details.

0.1.2.3 Stochastic Perturbations and Smooth Condition Numbers

Recall from previous sections that the condition number, of a computational problem with inputs $(\mathcal{X},\langle\cdot,\cdot\rangle_x)$ and outputs $(\mathcal{Y},\langle\cdot,\cdot\rangle_y)$, at $(x,y)\in\mathcal{V}\setminus\Sigma'$ is defined as:

$$\mu(x,y):=\max_{\substack{\dot{x}\in T_x\mathcal{X}\\ \|\dot{x}\|_x^2=1}}\|D\mathscr{S}(x,y)\dot{x}\|_y.$$

In many practical situations, there exists a discrepancy between worst case theoretical analysis and observed accuracy of an algorithm. There exist several approaches that attempt to rectify this discrepancy.

In many problems, the space of inputs has a much larger dimension than the one of the space of outputs ($m \gg n$). Then, it is natural to assume that infinitesimal perturbations of the input will produce drastic changes in the output only when they are performed in a few directions. Then, a possibly different approach to analyze accuracy of algorithms is to replace "worst direction" by a certain mean over all possible directions. This alternative was already suggested and studied in Weiss et al. Weiss *et al.* [1986] in the case of linear system solving $Ax = b$, and more generally, in Stewart [1990] in the case of matrix perturbation theory, where the first-order perturbation expansion is assumed to be random.

In *Appendix A*, following Armentano [2010], we extend this approach to a large class of computational problems, restricting ourselves to the case of directionally uniform perturbations.

Define the *pth-stochastic condition number* at (x, y) as:

$$\mu_{st}{}^{[p]}(x,y) := \left[\frac{1}{\mathrm{vol}(S_x^{m-1})}\int_{\dot{x}\in S_x^{m-1}} \|D\mathscr{S}(x)\dot{x}\|_y^p \, dS_x^{m-1}(\dot{x})\right]^{1/p}, \quad (p = 1, 2, \ldots),$$

where $\mathrm{vol}(S_x^{m-1}) = \frac{2\pi^{m/2}}{\Gamma(m/2)}$ is the measure of the unit sphere S_x^{m-1} in $T_x\mathfrak{X}$, and dS_x^{m-1} is the induced volume element. We will be mostly interested in the case $p = 2$, which we simply write μ_{st} and call it *stochastic condition number.*

Before stating the main theorem, we define the *Frobenius condition number* as:

$$\mu_F(x,y) := \|D\mathscr{S}(x)\|_F = \sqrt{\sigma_1^2 + \cdots + \sigma_n^2}$$

where $\|\cdot\|_F$ is the Frobenius norm and $\sigma_1, \ldots, \sigma_n$ are the singular values of the condition operator.

The main result in *Appendix A* is:

The pth-stochastic condition number satisfies

$$\mu_{st}{}^{[p]}(x,y) = \frac{1}{\sqrt{2}}\left[\frac{\Gamma\left(\frac{m}{2}\right)}{\Gamma\left(\frac{m+p}{2}\right)}\right]^{1/p} \cdot \mathbb{E}(\|\eta_{\sigma_1,\ldots,\sigma_n}\|^p)^{1/p},$$

where $\|\cdot\|$ *is the Euclidean norm in* $\mathbb{R}^n$ *and* $\eta_{\sigma_1,\ldots,\sigma_n}$ *is a centered Gaussian vector in* $\mathbb{R}^n$ *with diagonal covariance matrix* $Diag(\sigma_1^2, \ldots, \sigma_n^2)$*. In particular, for* $p = 2$

$$\mu_{st}(x, y) = \frac{\mu_F(x, y)}{\sqrt{m}}.$$

Since $\mu(x, y) \leq \mu_F(x, y) \leq \sqrt{n} \cdot \mu(x, y)$, we have from (A.1.3) that

$$\frac{1}{\sqrt{m}} \cdot \mu(x, y) \leq \mu_{st}(x, y) \leq \sqrt{\frac{n}{m}} \cdot \mu(x, y).$$

This result is most interesting when $m \gg n$, for in that case $\mu_{st}(x, y) \ll \mu(x, y)$. Thus, in these cases one may expect much better stability properties than those predicted by classical condition numbers.

In *Appendix A* we prove these results, extending them to the case of *pth-stochastic kth-componentwise condition numbers*. We also compute the stochastic condition number for different problems, namely, systems of linear equations, eigenvalue and eigenvector problems, finding kernels of linear transformations and solving polynomial systems of equations.

0.2 Random System of Equations

Let us consider a system of m polynomial equations in m unknowns over a field $\mathbb{K}$,

$$f_i(x) := \sum_{\|j\| \leq d_i} a_j^{(i)} x^j \quad (i = 1, \ldots, m).$$

The notation is the following: $x := (x_1, \ldots, x_m)$ denotes a point in $\mathbb{K}^m$, $j := (j_1, \ldots, j_m)$ a multi-index of non-negative integers, $\|j\| = \sum_{h=1}^{m} j_h$, $x^j = x^{j_1} \cdots x^{j_m}$, $a_j^{(i)} = a_{j_1,\ldots,j_m}^{(i)}$, and d_i is the degree of the polynomial f_i.

We are interested in the solutions of the system of equations

$$f_i(x) = 0 \quad (i = 1, \ldots, m),$$

lying in some subset V of $\mathbb{K}^m$. Throughout this second part, we are mainly concerned with the case $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$.

If we choose at random the coefficients $\{a_j^{(i)}\}$, then the solution of the system $f(x) = 0$, becomes a random subset of $\mathbb{K}^m$. This is the main object of this second part of this dissertation.

When we restrict to the case $\mathbb{K} = \mathbb{R}$, the simple and fundamental object to study is the number of solutions of the system lying in some Borel subset V of $\mathbb{R}^m$. Let us denote by $N^f(V)$ this number.

The study of the expectation of the number of real roots of a random polynomial started in the thirties with the work of Bloch & Pólya [1931]. Further investigations were made by Littlewood & Offord [1938]. However, the first sharp result is due to Kac [1943; 1949], who gives the asymptotic value

$$\mathbb{E}\left(N^f(\mathbb{R})\right) \approx \frac{2}{\pi}\log d, \quad \text{as} \quad d \to +\infty,$$

when the coefficients of the degree d univariate polynomial f are Gaussian centered independent random variables $N(0,1)$ (see the book by Bharucha-Reid & Sambandham [1986]).

The first important result in the study of real roots of random system of polynomial equations is due to Shub & Smale [1993b], where the authors computed the expectation of $N^f(\mathbb{R}^m)$ when the coefficients are Gaussian centered independent random variables having variances:

$$\mathbb{E}\left[(a_j^{(i)})^2\right] = \frac{d_i!}{j_1! \cdots j_m!\,(d_i - \|j\|)!}.$$

Their result was

$$\mathbb{E}\left(N^f(\mathbb{R}^m)\right) = \sqrt{d_1 \cdots d_m},$$

that is, the square root of the Bézout number associated to the system. The proof is based on a double fibration manipulation of the co-area formula. Some extensions of their work, including new results for one polynomial in one variable, can be found in Edelman & Kostlan [1995]. There are also other extensions to multi-homogeneous systems in McLennan [2002], and, partially, to sparse systems in Rojas [1996] and Malajovich & Rojas [2004]. A similar question for the number of critical points of real-valued polynomial random functions has been considered in Dedieu & Malajovich [2008].

The probability law of the Shub–Smale model defined above has the simplifying property of being invariant under the action of the orthogonal group in $\mathbb{R}^m$. In Kostlan [2002] one can find the classification of all Gaussian probability distributions over the coefficients with this geometric invariant property.

In 2005, Azaïs and Wschebor gave a new and deep insight to this problem. The key point is using the Rice formula for random Gaussian fields (cf. Azaïs & Wschebor [2009]). This formula allows one to extend the Shub–Smale result to other probability distributions over the coefficients. A general formula for $\mathbb{E}(N^f(V))$ when the random functions $f_i\,(i = 1, \dots, m)$ are stochastically independent and their law is centered and invariant under the orthogonal group on $\mathbb{R}^m$ can be found in Azaïs & Wschebor [2005]. This includes the Shub–Smale formula as a special case. Moreover, Rice formula appears to be the instrument to consider a major problem in the subject which is to find the asymptotic distribution of $N^f(V)$ (under some normalization). The only published results of which the author is aware concern asymptotic variances as $m \to +\infty$. (See Wschebor [2008] for a detailed description in this direction).

When the number of equations is less than the numbers of unknowns, generically, the set of solutions is a real algebraic variety of positive dimension. In this case, when the coefficients are taken at random, the description of the geometry becomes the main problem. In Bürgisser [2006; 2007] the expected value of certain parameters describing the geometry of this random algebraic variety are computed.

When we restrict to the case $\mathbb{K} = \mathbb{C}$ other interesting problems come into account, even for the case of one variable. For example,

How are the roots of complex random polynomials distributed?

The study of this question is one of the main research activities in the field of complex random polynomials. At the end of this dissertation we study the relation of this problem and the complexity of homotopy methods.

0.2.1 Main Contributions

In this section we introduce the main contributions given in this dissertation associated to random polynomials.

0.2.1.1 Random System of Polynomials over $\mathbb{R}$

In *Chapter 4* we recall some known results concerning the understanding of the set of solutions of random system of equations from Rice formulas point of view. Almost all results of this chapter are known however in this dissertation we develop a systematic way to analyze these problems with this powerful technic, hoping that this approach could be used to study other important problems related to the analysis of random algebraic varieties.

In *Chapter 4*, we begin giving an outline on Rice formulas for random fields. In the case of polynomial random fields we show the relation of Rice formulas with other technics to study the average number of solutions.

We also recall Shub-Smale result and we give a short proof of it based on Rice formulas.

At the end of this chapter we recall some known results about the undetermined case, that is, when the random system of equations has less equations than unknowns. More precisely, let us assume now that we have less equations than variables, that is, let $f : \mathbb{R}^n \to \mathbb{R}^k$ be a random system of polynomials such that $k < n$. In this case $\mathcal{Z}(f_1, \ldots, f_k) = f^{-1}(0)$ is a random algebraic variety of positive dimension. A natural questions come into account:

What is the average volume of $\mathcal{Z}(f_1, \ldots, f_k)$ *?*

At the end of *Chapter 4* we show how to attack this problem by means of the Rice formulas. In Bürgisser [2006] and Bürgisser [2007] one can find a nice study of this and other important questions concerning geometric properties of random algebraic varieties from a different point of view.

We will restrict ourselves to the particular case of the Shub-Smale distribution. Let us consider the random system of k homogeneous polynomial equations in

$m+1$ unknowns $f : \mathbb{R}^{m+1} \to \mathbb{R}^k$, given by

$$f_i(x) := \sum_{\|j\|=d_i} a_j^{(i)} x^j, \qquad (i = 1, \dots, k).$$

Assume that this system has the Shub-Smale distribution, that is, $\{a_j^{(i)}\}$ are Gaussian, centered, independent random variables having variances

$$\mathbb{E}\left[(a_j^{(i)})^2\right] = \binom{d_i}{j} = \frac{d_i!}{j_0! \cdots j_m!}.$$

Since f is homogeneous, we can restrict to the sphere $S^m \subset \mathbb{R}^{m+1}$ our study of the random set $\mathcal{Z}(f_1, \dots, f_k)$. Note that, generically, $\mathcal{Z}(f_1, \dots, f_k) \cap S^m$ is a smooth manifold of dimension $m-k$. Let us denote by λ_{m-k} the $m-k$ geometric measure.

Let $f : \mathbb{R}^{m+1} \to \mathbb{R}^k$ be the system defined above with the Shub-Smale distribution. Then, one has

$$\mathbb{E}(\lambda_{m-k}(\mathcal{Z}(f_1, \dots, f_k) \cap S^m)) = \sqrt{d_1 \cdots d_k}\, vol(S^{m-k+1}).$$

This result was first observed by Kostlan [1993] in the particular case $d_1 = \ldots = d_k$. We give a proof of this proposition based on the Rice formula for the geometric measure. We will see that the proof is almost the same as the proof of Shub-Smale result.

Furthermore, we will see how one can obtain another proof of this theorem from Shub-Smale result and the fairly known Crofton-Poincare formula of integral geometry.

Up to now all probability measures were introduced in a particular basis, namely, the monomial basis $\{x^j\}_{\|j\| \le d}$. However, in many situations, polynomial systems are expressed in different basis, such as, orthogonal polynomials, harmonic polynomials, Bernstein polynomials, etc. So, it is a natural question to ask:

What can be said about $N^f(V)$ when the randomization is performed in a different basis?

For the case of random orthogonal polynomials see Bharucha-Reid & Sambandham [1986], and Edelman & Kostlan [1995] for random harmonic polynomials.

In *Chapter 4* following Armentano & Dedieu [2009] we give an answer to the average number of real roots of a random system of equations expressed in the Bernstein basis. Let us be more precise:

The Bernstein basis is given by:

$$b_{d,k}(x) = \binom{d}{k} x^k (1-x)^{d-k}, \quad 0 \leq k \leq d,$$

in the case of univariate polynomials, and

$$b_{d,j}(x_1, \ldots, x_m) = \binom{d}{j} x_1^{j_1} \ldots x_m^{j_m} (1 - x_1 - \ldots - x_m)^{d - \|j\|}, \qquad \|j\| \leq d,$$

for polynomials in m variables, where $j = (j_1, \ldots, j_m)$ is a multi-integer, and $\binom{d}{j}$ is the multinomial coefficient.

Let us consider the set of real polynomial systems in m variables,

$$f_i(x_1, \ldots, x_m) = \sum_{\|j\| \leq d_i} a_j^{(i)} b_{d,j}(x_1, \ldots, x_m), \qquad (i = 1, \ldots, m).$$

Take the coefficients $a_j^{(i)}$ to be independent Gaussian standard random variables.

Define

$$\tau : \mathbb{R}^m \to \mathbb{P}\left(\mathbb{R}^{m+1}\right)$$

by

$$\tau(x_1, \ldots, x_m) = [x_1, \ldots, x_m, 1 - x_1 - \ldots - x_m].$$

Here $\mathbb{P}(\mathbb{R}^{m+1})$ is the projective space associated with $\mathbb{R}^{m+1}$, $[y]$ is the class of the vector $y \in \mathbb{R}^{m+1}$, $y \neq 0$, for the equivalence relation defining this projective space. The (unique) orthogonally invariant probability measure in $\mathbb{P}(\mathbb{R}^{m+1})$ is denoted by λ_m.

With the above notation the following result holds:

1. *For any Borel set V in $\mathbb{R}^m$ we have*

$$\mathbb{E}\left(N^f(V)\right) = \lambda_m(\tau(V))\sqrt{d_1 \dots d_m}.$$

In particular

2. $\mathbb{E}\left(N^f\right) = \sqrt{d_1 \dots d_m}$,
3. $\mathbb{E}\left(N^f(\Delta^m)\right) = \sqrt{d_1 \dots d_m}/2^m$, *where*

$$\Delta^m = \{x \in \mathbb{R}^m \ : \ x_i \geq 0 \text{ and } x_1 + \dots + x_m \leq 1\},$$

4. *When $m = 1$, for any interval $I = [\alpha, \beta] \subset \mathbb{R}$, one has*

$$\mathbb{E}\left(N^f(I)\right) = \frac{\sqrt{d}}{\pi}\left(\arctan(2\beta - 1) - \arctan(2\alpha - 1)\right).$$

Moreover, in *Chapter 4* we extend last result on Bernstein polynomial systems. We give a general formula to compute the expected number of roots of some random systems of equations.

Let $U \subset \mathbb{R}^m$ be an open subset, and let $\varphi_0, \dots, \varphi_m : U \to \mathbb{R}$ be $(m+1)$ differentiable functions. Assume that, for every $x \in U$, the values $\varphi_i(x)$ do not vanish at the same time. Then we can define the map $\Lambda : U \to \mathbb{P}(\mathbb{R}^{m+1})$ by $\Lambda(x) = [\varphi_0(x), \dots, \varphi_m(x)]$.

Let f be the system of m-equations in m real variables

$$f_i(x_1, \dots, x_m) := \sum_{\|j\| = d_i} a_j^{(i)} \varphi_0(x)^{j_0} \cdots \varphi_m(x)^{j_m}, \qquad (i = 1, \dots, m),$$

where $x = (x_1, \dots, x_m) \in U$.

We denote by $N^f(U)$ the number of roots of the system of equations $f_i(x) = 0$, $(i = 1, \dots, m)$ lying in U.

Then,

Let f be the system of equations given above, where the $\{a_j^{(i)}\}$ are independent Gaussian centered random variables with variance $\binom{d_i}{j}$.

Then,

$$\mathbb{E}\left[N^f(U)\right] = \frac{\sqrt{d_1 \cdots d_m}}{vol(\mathbb{P}(\mathbb{R}^{m+1}))} \int_{z \in \mathbb{P}(\mathbb{R}^{m+1})} \# \Lambda^{-1}(\{z\}) \, dz.$$

where $\# \emptyset = 0$.

We will see how this result extend the previous result on Bernstein polynomials and we also show some simple non-polynomial examples.

0.2.1.2 Random System of Polynomials over $\mathbb{C}$

In *Chapter 5* we study complex random systems of polynomial equations. The main objective is to introduce the technics of Rice formulas in the realm complex random fields. At the end we give a probabilistic approach of Bézout's theorem using Rice Formulas.

This chapter follows closely a joint work under construction with Federico Dalmao and Mario Wschebor [Armentano *et al.*, 2012]. The main objective of this work is to give a probabilistic proof of Bézout's theorem. More precisely:

Assume that f has the complex analogue of Shub-Smale distribution and denote by N the number of projective zeros of f. Then,

$$N = \mathcal{D} \quad \textit{almost surely,}$$

where $\mathcal{D} = \prod_{\ell=1}^{m} d_i$ *is Bézout number.*

The proof we have attempted was divided into two steps:
- First prove that the expected value of N is $\mathcal{D}$;
- Secondly, prove that the variance of the random variable $N - \mathcal{D}$ is zero.

Both steps can be analyzed with Rice formulas. The first step follows similarly to the proof of Shub-Smale result for the real case, and is even much simpler. For the second step we use a version of the Rice formula for the k-moment.

The second step involves many computations. Even though we could not finish the proof of the second step, we will show how to proceed in the computations and we will show the main difficulties. On the particular case of $m = 1$, that is, the Fundamental Theorem of Algebra, we finish the proof.

0.2.1.3 Fekete Points and Random Polynomials

In *Chapter 6*, following Armentano *et al.* [2011], we see that points in the sphere associated with roots of Shub-Smale complex analogue random polynomials via the stereographic projection, are surprisingly well-suited with respect to the minimal logarithmic energy on the sphere. That is, they provide a fairly good approximation to a classical minimization problem over the sphere, namely, the Elliptic Fekete points problem.

Let us be more precise.

Given $x_1, \ldots, x_N \in \mathbb{S}^2 = \{x \in \mathbb{R}^3 : \|x\| = 1\}$, let

$$V(x_1, \ldots, x_N) = \ln \prod_{1 \leq i < j \leq N} \frac{1}{\|x_i - x_j\|} = - \sum_{1 \leq i < j \leq N} \ln \|x_i - x_j\|$$

be the logarithmic energy of the N-tuple $x_1, \ldots, x_N$.

Let

$$V_N = \min_{x_1, \ldots, x_N \in \mathbb{S}^2} V(x_1, \ldots, x_N)$$

denote the minimum of this function. N-tuples minimizing the quantity V are usually called Elliptic Fekete Points. The problem of finding (or even approximate) such optimal configurations is a classical problem (see Whyte [1952] for its origins).

During the last decades this problem has attracted much attention, and the number of papers concerning it has grown amazingly. The reader may see Kuijlaars & Saff [1998] for a nice survey.

In the list of Smale's problems for the XXI Century Smale [2000], problem number 7 reads:

Can one find $x_1, \ldots, x_N \in \mathbb{S}^2$ such that

$$V(x_1, \ldots, x_N) - V_N \leq c \ln N,$$

c a universal constant?

More precisely, Smale demands a real number algorithm in the sense of Blum *et al.* [1998] that with input N returns a N-tuple $x_1, \ldots, x_N$ satisfying last inequality, and such that the running time is polynomial on N.

One of the main difficulties when dealing with this problem is that the value of V_N is not even known up to logarithmic precision. In Rakhmanov *et al.* [1994] the authors proved that if one defines C_N by

$$(*) \qquad V_N = -\frac{N^2}{4}\ln\left(\frac{4}{e}\right) - \frac{N\ln N}{4} + C_N N,$$

then,

$$-0.112768770... \leq \liminf_{N\to\infty} C_N \leq \limsup_{N\to\infty} C_N \leq -0.0234973...$$

Let $X_1, \ldots, X_N$ be independent random variables with common uniform distribution over the sphere. One can easily show that the expected value of the function $V(X_1, \ldots, X_N)$ in this case is,

$$(**) \qquad \mathbb{E}(V(X_1, \ldots, X_N)) = -\frac{N^2}{4}\ln\left(\frac{4}{e}\right) + \frac{N}{4}\ln\left(\frac{4}{e}\right).$$

Thus, this random choice of points in the sphere with independent uniform distribution already provides a reasonable approach to the minimal value V_N, accurate to the order of $O(N\ln N)$.

On one side, this probability distribution has an important property, namely, invariance under the action of the orthogonal group on the sphere. However, on the other hand this probability distribution lacks on correlation between points. More precisely, in order to obtain well-suited configurations one needs some kind of repelling property between points, and in this direction independence is not favorable. Hence, it is a natural question whether other handy orthogonally invariant probability distributions may yield better expected values. Here is where complex random polynomials comes into account!

Given $z \in \mathbb{C}$, let

$$\hat{z} := \frac{(z,1)}{1+|z|^2} \in \mathbb{C}\times\mathbb{R} \cong \mathbb{R}^3$$

be the associated points in the Riemann Sphere, i.e. the sphere of radius $1/2$ centered at $(0,0,1/2)$. Finally, let

$$X = 2\hat{z} - (0,0,1) \in \mathbb{S}^2$$

be the associated points in the unit sphere.

Given a polynomial f in one complex variable of degree N, we consider the mapping

$$f \mapsto V(X_1, \ldots, X_N),$$

where X_i $(i = 1, \ldots, N)$ are the associated roots of f in the unit sphere. Notice that this map is well defined in the sense that it does not depend on the way we choose to order the roots.

The main contribution of *Chapter 6* is the following:

> *Let $f(z) = \sum_{k=0}^{N} a_k z^k$ be a complex random polynomial, such that the coefficients a_k are independent complex random variables, such that the real and imaginary parts of a_k are independent (real) Gaussian random variables centered at* 0 *with variance $\binom{N}{k}$. Then, with the notations above,*

$$\mathbb{E}\left(V(X_1, \ldots, X_N)\right) = -\frac{N^2}{4} \ln\left(\frac{4}{e}\right) - \frac{N \ln N}{4} + \frac{N}{4} \ln \frac{4}{e}.$$

Comparing this result with equations (*) and (**), we see that the value of V is surpringsingly small at points coming from the solution set of this random polynomials. More precisely, necessarily many random realizations of the coefficients will produce values of V below the average and very close to V_N, possibly close enough to satisfy the inequality in Smale's 7th problem.

Notice that, taking the homogeneous counterpart of f, our main result can be restated for random homogeneous polynomials and considering its complex projective solutions, under the identification of $\mathbb{P}(\mathbb{C}^2)$ with the Riemann sphere. In this fashion, the induced probability distribution over the space of homogeneous polynomials in two complex variables corresponds to the classical unitarily invariant Hermitian structure of the respective space (see Blum *et al.* [1998]). Therefore, the probability distribution of the roots in $\mathbb{P}(\mathbb{C}^2)$ is invariant under the action of the unitary group.

It is not difficult to prove that the unitary group action over $\mathbb{P}(\mathbb{C}^2)$ correspond to the special orthogonal group of the unit sphere. Hence, the distribution of the associated random roots on the sphere is orthogonally invariant. Thus, our main result is another geometric confirmation of the repelling property of the roots of this Gaussian random polynomials.

Part of the motivation of 7th Problem of Smale is the search for a polynomial all of whose roots are well conditioned, in the context of Shub & Smale [1993c].

Shub & Smale [1993b] proved that well-conditioned polynomials are highly probable. In Shub & Smale [1993c] the problem was raised as to how to write a deterministic algorithm which produces a polynomial g all of whose roots are well-conditioned. It was also realized that a polynomial whose projective roots (seen as points in the Riemann sphere) have logarithmic energy close to the minimum as in Smale's problem after scaling to $\mathbb{S}^2$, are well conditioned.

From the point of view of Shub & Smale [1993c], the ability to choose points at random already solves the problem. Here, instead of trying to use the logarithmic energy function $V(\cdot)$ to produce well-conditioned polynomials, we use the fact that random polynomials are well-conditioned, to try to produce low-energy N-tuples.

Part I

Complexity of Path-Following Methods

Chapter 1

Complexity of The Eigenvalue Problem I: Geodesics in the Condition Metric

In this chapter we study path-following methods for the eigenvalue problem. We introduce a projective framework to analyze this problem. We define a condition number and a Newton's map appropriate for this context, proving a version of the γ-Theorem. The main result of this chapter is to bound the complexity of path-following methods in terms of the length of the path in the condition metric.

This chapter follows closely Armentano [2011a].

1.1 Introduction and Main Results

1.1.1 Introduction

In this chapter we study the complexity of path-following methods to solve the eigenvalue problem:

$$(\lambda I_n - A)v = 0, \quad v \neq 0,$$

where $A \in \mathbb{K}^{n\times n}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$), $v \in \mathbb{K}^n$, $\lambda \in \mathbb{K}$. Classical algorithms for solving the eigenvalue problem may be divided into two classes: QR methods (including Hessenberg reduction, single or double shift strategy, deflation), and Krylov subspace methods; see Wilkinson [1965], Golub & Van Loan [1996], Stewart [2001], Watkins [2007].

Surprisingly, the complexity of the eigenvalue problem is still an open problem. It may be formulated in the following terms: given an algorithm designed to solve the eigenvalue problem,

1. For which class of matrices does it converge ?

2. What is the average number of steps, in a given probabilistic model on the set of inputs, to obtain a given accuracy on the output ?

The two following examples show that such questions are particularly difficult:

- QR algorithm with Wilkinson's single shift diverges for a non-empty open set of matrices (see Batterson & Smillie [1989a;b]).

- QR algorithm is convergent for almost every complex matrix. However, even for the choice of *Gaussian Orthogonal Ensemble*, as a probabilistic model, question (2) remains unanswered (see Deift [2008]).

In this chapter we consider the eigenvalue problem as a bilinear polynomial system of equations and we consider homotopy methods to solve it. The system $(\lambda I_n - A)v = 0$, $v \neq 0$, is the endpoint of a path of problems

$$(\lambda(t)I_n - A(t))v(t) = 0, \ v(t) \neq 0, \ 0 \leq t \leq 1,$$

with $(A(1), \lambda(1), v(1)) = (A, \lambda, v)$. Starting from a known triple $(A(0), \lambda(0), v(0))$ we "follow" this path to reach the target system $(\lambda I_n - A)v = 0$. The algorithmic way to do so is to construct a finite number of triples

$$(A_k, \lambda_k, v_k), \ 0 \leq k \leq K,$$

with $A_k = A(t_k)$, and $0 = t_0 < t_1 < \ldots < t_K = 1$, and where λ_k, v_k are approximations of $\lambda(t_k)$, $v(t_k)$. The complexity of this algorithm (defined more precisely below) is measured by the number K of steps sufficient to validate this approximation. In this chapter we relate K with a geometric invariant, namely, the **condition length** of the path.

We begin with the geometric framework of our problem. Since the eigenvalue problem is homogeneous in $v \in \mathbb{K}^n$ and also in $(A, \lambda) \in \mathbb{K}^{n\times n} \times \mathbb{K}$, we define the solution variety as

$$\mathcal{V} =: \left\{(A, \lambda, v) \in \mathbb{P}\left(\mathbb{K}^{n\times n} \times \mathbb{K}\right) \times \mathbb{P}\left(\mathbb{K}^n\right) : \ (\lambda I_n - A)v = 0\right\},$$

where $\mathbb{P}(\mathbb{E})$ denotes the projective space associated with the vector space $\mathbb{E}$. We speak interchangeably of a non zero vector and its corresponding class in the projective space.

Note that the solution variety $\mathcal{V}$ differs from the definition given *Subsection 0.1.1.*

1.1.2 A Bihomogeneous Newton's Method

Given a non-zero matrix $A \in \mathbb{K}^{n\times n}$, we define the evaluation map $F_A : \mathbb{K} \times \mathbb{K}^n \to \mathbb{K}^n$, by

$$F_A(\lambda, v) := (\lambda I_n - A)v.$$

Associated to F_A we define $N_A : \mathbb{K} \times (\mathbb{K}^n - \{0\}) \to \mathbb{K} \times \mathbb{K}^n$, given by

$$N_A(\lambda, v) := (\lambda, v) - \left(DF_A(\lambda, v)|_{\mathbb{K}\times v^\perp}\right)^{-1} F_A(\lambda, v), \tag{1.1.1}$$

defined for all (λ, v) such that $DF_A(\lambda, v)|_{\mathbb{K}\times v^\perp}$ is surjective. Here $v^\perp$ is the Hermitian complement of v in $\mathbb{K}^n$. This map is homogeneous of degree 1 in v, therefore, N_A induces a map from $\mathbb{K} \times \mathbb{P}(\mathbb{K}^n)$ into itself.

We define the *Newton map* N on $(\mathbb{K}^{n\times n} - \{0_n\}) \times \mathbb{K} \times (\mathbb{K}^n - \{0\})$ by

$$N(A, \lambda, v) := \big(A, N_A(\lambda, v)\big).$$

This map N is a bihomogeneous map of degree 1 in (A, λ) and v. Hence N is well-defined on $\mathbb{P}\big(\mathbb{K}^{n\times n} \times \mathbb{K}\big) \times \mathbb{P}(\mathbb{K}^n)$ (see *Section 1.4*).

Given $A \in \mathbb{K}^{n\times n}$, $A \neq 0_n$, and $(\lambda_0, v_0) \in \mathbb{K} \times \mathbb{K}^n$, $v_0 \neq 0$, the *Newton sequence* associated to A is defined by

$$(A, \lambda_{k+1}, v_{k+1}) := N(A, \lambda_k, v_k), \quad k \geq 0.$$

We say that this sequence *converges immediately quadratically* to a solution of the eigenvalue problem $(A, \lambda, v) \in \mathcal{V}$ when

$$d_{\mathbb{P}^2}\left((A, \lambda_k, v_k), (A, \lambda, v)\right) \leq \left(\frac{1}{2}\right)^{2^k-1} \cdot d_{\mathbb{P}^2}\left((A, \lambda_0, v_0), (A, \lambda, v)\right),$$

for all positive integer k. Here $d_{\mathbb{P}^2}(\cdot, \cdot)$ is the induced Riemannian distance on $\mathbb{P}\big(\mathbb{K}^{n\times n} \times \mathbb{K}\big) \times \mathbb{P}(\mathbb{K}^n)$ (see *Section 1.2.1*). In that case we say that (A, λ_0, v_0) is an *approximate solution* of the eigenvalue problem $(A, \lambda, v) \in \mathcal{V}$.

1.1.3 The Predictor-Corrector Algorithm

Let $\Gamma(t) = (A(t), \lambda(t), v(t))$, $a \leq t \leq b$, be a representative path in $\mathcal{V}$. To approximate the path Γ by a finite sequence we use the following predictor-corrector strategy: given a mesh $a = t_0 < t_1 < \ldots < t_K = b$ and a triple $(A(t_0), \lambda_0, v_0) \in \mathbb{K}^{n\times n} \times \mathbb{K} \times \mathbb{K}^n$, $(v_0 \neq 0)$, we define

$$(A(t_{k+1}), \lambda_{k+1}, v_{k+1}) := N(A(t_{k+1}), \lambda_k, v_k), \quad 0 \leq k \leq K-1,$$

(in case the Newton map is defined). We say that the sequence $(A(t_k), \lambda_k, v_k)$, $0 \leq k \leq K$, *approximates* the path $\Gamma(t)$, $a \leq t \leq b$, when, for any $k = 0, \ldots, K$, $(A(t_k), \lambda_k, v_k)$ is an approximate solution of the eigentriple $\Gamma(t_k) = (A(t_k), \lambda(t_k), v(t_k)) \in \mathcal{V}$. In that case we define the *complexity of the sequence* by K.

1.1.4 Condition of a Triple and Condition Length

Let $\mathcal{W} \subset \mathcal{V}$ be the set of *well-posed* problems, that is the set of triples $(A, \lambda, v) \in \mathcal{V}$ such that λ is a simple eigenvalue (see *Section 1.3*). In that case, for a fixed representative $(A, \lambda, v) \in \mathcal{V}$, the operator $\Pi_{v^\perp}(\lambda I_n - A)|_{v^\perp}$ is invertible, where $\Pi_{v^\perp}$ denotes the orthogonal projection of $\mathbb{K}^n$ onto $v^\perp$. The condition number of (A, λ, v) is defined by

$$\mu(A, \lambda, v) := \max\left\{1, \|A\|_F \cdot \|\Pi_{v^\perp}(\lambda I_n - A)|_{v^\perp}{}^{-1}\|\right\}, \qquad (1.1.2)$$

where $\|\cdot\|_F$ and $\|\cdot\|$ are the Frobenius and operator norms in the space of matrices. We also let $\mu(A,\lambda,v)=\infty$ when $(A,\lambda,v)\in\mathcal{V}-\mathcal{W}$; (see *Section 1.3*).

When $\Gamma(t)$, $a\leq t\leq b$, is an absolutely continuous path in $\mathcal{W}$, we define its *condition-length* as

$$\ell_\mu(\Gamma) := \int_a^b \left\|\dot{\Gamma}(t)\right\|_{\Gamma(t)} \cdot \mu\left(\Gamma(t)\right)\, dt, \tag{1.1.3}$$

where $\left\|\dot{\Gamma}(t)\right\|_{\Gamma(t)}$ is the norm of $\dot{\Gamma}(t)$ in the unitarily invariant Riemannian structure on $\mathcal{V}$ (see *Section* 1.2.1).

1.1.5 Main Results

The main theorem concerning the convergence of Newton's iteration is the following.

Theorem 1. *There is a universal constant $u_0>0$ with the following property. Let (A,λ,v), $(A,\lambda_0,v_0)\in\mathbb{P}\big(\mathbb{K}^{n\times n}\times\mathbb{K}\big)\times\mathbb{P}(\mathbb{K}^n)$. If $(A,\lambda,v)\in\mathcal{W}$ and*

$$d_{\mathbb{P}^2}\left((A,\lambda_0,v_0),(A,\lambda,v)\right) < \frac{u_0}{\mu(A,\lambda,v)},$$

then, (A,λ_0,v_0) is an approximate solution of (A,λ,v).
(One may choose $u_0=0.0739$).

Theorem 1 is a version of the so called γ-*theorem* (see Blum *et al.* [1998]), which gives the size of the basin of attraction of Newton's method. Different versions of the γ-theorem for the symmetric eigenvalue problem and for the generalized eigenvalue problem are given in Dedieu [2006] and Dedieu & Shub [2000] respectively.

Theorem 1 is the main ingredient to prove complexity results for path-following methods.

The proof of *Theorem 1* follows from a version of the γ-theorem for the map $N_A:\mathbb{K}\times\mathbb{P}(\mathbb{K}^n)\to\mathbb{K}\times\mathbb{P}(\mathbb{K}^n)$ which is interesting in itself (see *Section 1.4*).

Following these lines our main result is:

Theorem 2. *There is a universal constant $C > 0$ such that for any absolutely continuous path Γ in $\mathcal{W}$, there exists a sequence which approximates Γ, and such that the complexity of the sequence is*

$$K \leq C\,\ell_{\mu}(\Gamma) + 1.$$

(One may choose $C = 120$).

The proof of *Theorem 2* is given in *Section 1.5*.

1.1.6 Comments

In their seminal paper [Shub & Smale, 1993a], Shub and Smale relate, in the context of polynomial system solving, the complexity K to three ingredients: the degree of the considered system, the length of the path $\Gamma(t)$, and the condition number of the path. Precisely, they obtain the complexity

$$K \leq C D^{3/2} \ell(\Gamma)\mu(\Gamma)^2),$$

where C is a universal constant, D is the degree of the system, $\ell(\Gamma)$ is the length of Γ in the associated Riemannian structure, and $\mu(\Gamma) = \sup_{a \leq t \leq b} \mu\left(\Gamma(t)\right)$. Similar results for the generalized eigenvalue problem were obtained in Dedieu & Shub [2000].

In Shub [2009] the complexity K of path-following methods for the polynomial system solving problem is analyzed in terms of the condition length of the path.

In the context of polynomial system solving, the eigenvalue problem may be considered as a quadratic system of equations. However, Shub & Smale [1993a] and Shub & Smale [1996] do not apply since the eigenvalue problem as a quadratic system belongs to the subset of ill-posed problems of generic quadratic systems. (See Li [1997]). Therefore, in order to analyze the complexity of the eigenvalue problem, a different framework is required. Here we consider the eigenvalue problem as a bilinear problem (see *Subsection 1.2.2.1*).

The approach considered in this chapter is greatly inspired by Shub [2009].

Note: Throughout this chapter we will work with $\mathbb{K} = \mathbb{C}$. However most definitions and results can be extended immediately to the case $\mathbb{K} = \mathbb{R}$. Whenever it is necessary we shall state the difference.

1.2 Riemannian Structures and the Solution Variety

In this section we define the canonical metric structures and study some basic topological and algebraic properties of the solution variety for the eigenvalue problem.

1.2.1 Canonical Metric Structures

The space $\mathbb{K}^n$ is equipped with the canonical Hermitian inner product $\langle \cdot, \cdot \rangle$. The space $\mathbb{K}^{n\times n}$ is equipped with the Frobenius Hermitian inner product

$$\langle A, B \rangle_F := \text{trace } (B^* A),$$

where B^* denotes the adjoint of B.

In general, if $\mathbb{E}$ is a finite dimensional vector space over $\mathbb{K}$ with the Hermitian inner product $\langle \cdot, \cdot \rangle$, we can define an Hermitian structure over $\mathbb{P}(\mathbb{E})$ in the following way: for $x \in \mathbb{E}$,

$$\langle w, w' \rangle_x := \frac{\langle w, w' \rangle}{\|x\|^2},$$

for all w, w' in the Hermitian complement $x^\perp$ of x in $\mathbb{E}$, which is a natural representative of the tangent space $T_x\mathbb{P}(\mathbb{E})$. Let $d_{\mathbb{P}}(x, y)$ be the angle between the vectors x and y.

The space $\mathbb{P}\big(\mathbb{K}^{n\times n} \times \mathbb{K}\big) \times \mathbb{P}(\mathbb{K}^n)$ inherits the Hermitian product structure

$$\|(\dot{A}, \dot{\lambda}, \dot{v})\|^2_{(A,\lambda,v)} = \|(\dot{A}, \dot{\lambda})\|^2_{(A,\lambda)} + \|\dot{v}\|^2_v \tag{1.2.1}$$

for all $(\dot{A}, \dot{\lambda}, \dot{v}) \in (A, \lambda)^\perp \times v^\perp$.

We denote by $d_{\mathbb{P}^2}(\cdot, \cdot)$ the induced Riemannian distance on $\mathbb{P}\big(\mathbb{K}^{n\times n} \times \mathbb{K}\big) \times \mathbb{P}(\mathbb{K}^n)$.

Throughout this chapter we denote by the same symbol $d_{\mathbb{P}}$ distances over $\mathbb{P}(\mathbb{K}^n)$, $\mathbb{P}(\mathbb{K}^{n\times n})$ and $\mathbb{P}(\mathbb{K}^{n\times n}\times\mathbb{K})$.

1.2.2 The Solution Variety $\mathcal{V}$

Recall that the solution variety $\mathcal{V}\subset\mathbb{P}\big(\mathbb{K}^{n\times n}\times\mathbb{K}\big)\times\mathbb{P}(\mathbb{K}^n)$ is given by the set of triples (A,λ,v) such that $(\lambda I_n-A)v=0$. Note that $\mathcal{V}$ is the set of equivalence classes of the set $\{F=0\}$, where $F:\mathbb{K}^{n\times n}\times\mathbb{K}\times(\mathbb{K}^n-\{0\})\to\mathbb{K}^n$ is the multihomogenous system of polynomials given by

$$F(A,\lambda,v)=(\lambda I_n-A)v. \tag{1.2.2}$$

Therefore $\mathcal{V}$ is an algebraic subvariety of the product $\mathbb{P}\big(\mathbb{K}^{n\times n}\times\mathbb{K}\big)\times\mathbb{P}(\mathbb{K}^n)$. Moreover, since 0 is a regular value of F, we conclude that $\mathcal{V}$ is also a smooth submanifold of $\mathbb{P}\big(\mathbb{K}^{n\times n}\times\mathbb{K}\big)\times\mathbb{P}(\mathbb{K}^n)$. Its dimension over $\mathbb{K}$ is given by

$$\dim\mathcal{V}=\dim(\mathbb{K}^{n\times n}\times\mathbb{K}\times\mathbb{K}^n)-n-2=n^2-1.$$

The tangent space $T_{(A,\lambda,v)}\mathcal{V}$ to $\mathcal{V}$ at (A,λ,v) is the set of triples

$$(\dot A,\dot\lambda,\dot v)\in\mathbb{K}^{n\times n}\times\mathbb{K}\times\mathbb{K}^n,$$

satisfying

$$(\dot\lambda I_n-\dot A)v+(\lambda I_n-A)\dot v=0;\quad \langle\dot A,A\rangle_F+\dot\lambda\overline{\lambda}=0;\quad \langle\dot v,v\rangle=0. \tag{1.2.3}$$

Remark 1.2.1. The solution variety $\mathcal{V}$ inherits the Hermitian structure from $\mathbb{P}\big(\mathbb{K}^{n\times n}\times\mathbb{K}\big)\times\mathbb{P}(\mathbb{K}^n)$ defined in (1.2.1).

We denote by π_1 and π_2 the restriction to $\mathcal{V}$ of the canoncial projections onto $\mathbb{P}\big(\mathbb{K}^{n\times n}\times\mathbb{K}\big)$ and $\mathbb{P}(\mathbb{K}^n)$ respectively.

Note that $\pi_1(\mathcal{V})\subset\mathbb{P}\big(\mathbb{K}^{n\times n}\times\mathbb{K}\big)$ does not include the pair $(0_n,1)$. Therefore we can define the map

$$\pi:\mathcal{V}\to\mathbb{P}(\mathbb{K}^{n\times n}),\quad \pi:=p\circ\pi_1,$$

where p is the canonical projection

$$p : \mathbb{P}\big(\mathbb{K}^{n\times n} \times \mathbb{K}\big) - \{(0_n, 1)\} \to \mathbb{P}(\mathbb{K}^{n\times n}), \quad p(A, \lambda) = A. \tag{1.2.4}$$

$$\begin{tikzcd}
 & \mathcal{V} \arrow[dl, "\pi_1"'] \arrow[dd, "\pi"] \arrow[dr, "\pi_2"] & \\
\mathbb{P}\big(\mathbb{K}^{n\times n} \times \mathbb{K}\big) - \{(0_n, 1)\} \arrow[dr, "p"'] & & \mathbb{P}(\mathbb{K}^n) \\
 & \mathbb{P}(\mathbb{K}^{n\times n}) &
\end{tikzcd}$$

The derivative

$$D\pi(A, \lambda, v) : T_{(A,\lambda,v)}\mathcal{V} \to T_A\mathbb{P}(\mathbb{K}^{n\times n}), \tag{1.2.5}$$

is a linear operator between spaces of equal dimension.

Definition 1. We say that the triple $(A, \lambda, v) \in \mathcal{V}$ is *well-posed* when $D\pi(A, \lambda, v)$ is an isomorphism. Let $\mathcal{W}$ be the set of well-posed triples, and $\Sigma' := \mathcal{V} - \mathcal{W}$ be the *ill-posed variety*. Let $\Sigma = \pi(\Sigma') \subset \mathbb{P}(\mathbb{K}^{n\times n})$ be the *discriminant variety*, i.e. the subset of ill-posed inputs.

Lemma 1.2.1. *Σ' is the set of triples $(A, \lambda, v) \in \mathcal{V}$ such that λ is not a simple eigenvalue.*

Proof. The linear operator (1.2.5) is given by

$$D\pi(A, \lambda, v)(\dot{A}, \dot{\lambda}, \dot{v}) = \dot{A} + \frac{\dot{\lambda} \cdot \overline{\lambda}}{\|A\|_F^2} \cdot A, \qquad (\dot{A}, \dot{\lambda}, \dot{v}) \in T_{(A,\lambda,v)}\mathcal{V}.$$

According to (1.2.3), a non-trivial triple in the kernel of $D\pi(A, \lambda, v)$ has the form $(\frac{-\dot{\lambda}\cdot\overline{\lambda}}{\|A\|_F^2} A, \dot{\lambda}, \dot{v})$, where $\langle \dot{v}, v\rangle = 0$, $\dot{v} \neq 0$, and

$$\dot{\lambda}\left(1 + \frac{|\lambda|^2}{\|A\|_F^2}\right) v + (\lambda I_n - A)\dot{v} = 0.$$

Then, $\operatorname{rank}[(\lambda I_n - A)^2] < n - 1$. $\square$

Corollary 1. *Σ' is an algebraic subvariety of $\mathcal{V}$.*

Remark 1.2.2. When $\mathbb{K} = \mathbb{C}$, by the *Main Therorem of elimination theory* (cf. [Mumford, 1976, pp. 33]) and the fact that the projection p is Zariski-closed (cf. [Mumford, 1976, Corollary 2.28]), we conclude from *Corollary 1* that Σ is an algebraic subvariety of $\mathbb{P}(\mathbb{K}^{n\times n})$.

Remark 1.2.3. The solution variety $\mathcal{V}$ is connected since each $(A, \lambda, v) \in \mathcal{V}$ can be connected by a path (in $\mathcal{V}$) with a triple of the form $(vv^*, \|v\|^2, v) \in \mathcal{V}$. Here v^* is the conjugate transpose of v.

Lemma 1.2.2. *(i) When $\mathbb{K} = \mathbb{C}$, $\mathcal{W}$ is connected.*

(ii) When $\mathbb{K} = \mathbb{R}$, $\mathcal{W}$ has two connected components.

Proof. (i) Since $\mathcal{V}$ is connected, the result follows from *Corollary 1* and the fact that a complex algebraic subvariety of $\mathcal{V}$ cannot disconnect it (cf. [Blum *et al.*, 1998, pp. 196]).
(ii) It is enough to prove the lemma in the affine case. Let $\hat{\mathcal{V}}$ and $\hat{\mathcal{W}}$ be the affine spaces associated to $\mathcal{V}$ and $\mathcal{W}$. Let $\varphi : \hat{\mathcal{V}} \to \mathbb{R}^n \times \mathbb{R}^{n\times n}$ be the continuous map given by $\varphi(A, \lambda, v) = (v, \lambda I_n - A)$. Define the subsets

$$\mathcal{L} := \{(w, M) \in \mathbb{R}^n \times \mathbb{R}^{n\times n} : \; Mw = 0\}$$

and

$$\mathcal{B} := \{(w, M) \in \mathcal{L} : \mathrm{rank}(M + ww^T) = n\}.$$

Note that φ projects $\hat{\mathcal{V}}$ onto $\mathcal{L}$, and therefore $\mathcal{L}$ is connected. Moreover $\varphi(\hat{\mathcal{W}}) = \mathcal{B}$. The second assertion follows from the fact that $\Pi_{w^\perp} M|_{w^\perp} = \Pi_{w^\perp}(M + ww^T)|_{w^\perp}$, for all $(w, M) \in \mathcal{L}$.
Note that, for all $(w, M) \in \mathcal{B}$, $\varphi^{-1}(w, M) = \{(M + \alpha I_n, \alpha, w) : \; \alpha \in \mathbb{R}\}$ is a one dimensional subspace of $\hat{\mathcal{W}}$. Therefore, the set $\hat{\mathcal{B}} := \{(M, 0, w) : \; (w, M) \in \mathcal{B}\}$ is a deformation retract of $\hat{\mathcal{W}}$. It is clear that $\hat{\mathcal{B}}$ and $\mathcal{B}$ are homeomorphic.

Then, the lemma follows from the fact that $\mathcal{B}$ has two connected component on $\mathcal{L}$. $\square$

1.2.2.1 Multidegree of $\mathcal{V}$

In this item we will see that the bilinear approach considered in this chapter gives the correct number of roots of the eigenvalue problem.

For the sake of simplicity in the exposition, we will restrict ourself to the case $\mathbb{K} = \mathbb{C}$. This subsection follows closely D'Andrea *et al.* [2011].

Since $\mathcal{V}$ is an algebraic subvariety of the product space $\mathbb{P}\big(\mathbb{C}^{n\times n} \times \mathbb{C}\big) \times \mathbb{P}(\mathbb{C}^n)$, there is a natural algebraic invariant associated to $\mathcal{V}$, namely, the *multidegree* of $\mathcal{V}$. This invariant is given by the numbers $\deg_{(n^2-1-i,i)}(\mathcal{V})$, $i = 0, \ldots, n-1$, where $\deg_{(n^2-1-i,i)}(\mathcal{V})$ is the number of points of intersection of $\mathcal{V}$ with the product $\Lambda \times \Lambda' \subset \mathbb{P}\big(\mathbb{C}^{n\times n} \times \mathbb{C}\big) \times \mathbb{P}(\mathbb{C}^n)$, where $\Lambda \subset \mathbb{P}\big(\mathbb{C}^{n\times n} \times \mathbb{C}\big)$ and $\Lambda' \subset \mathbb{P}(\mathbb{C}^n)$ are generic (n^2-1-i)-codimension plane and i-codimension plane respectively (see Fulton [1984]).

Lemma 1.2.3. $\deg_{(n^2-1-i,i)}(\mathcal{V}) = \binom{n}{i+1}$ *for* $i = 0, \ldots, n-1$.

In order to give a proof of this lemma we recall some definitions from *intersection theory* (cf. Fulton [1984]). (See also D'Andrea *et al.* [2011]).

The Chow ring of $\mathbb{P}\big(\mathbb{C}^{n\times n} \times \mathbb{C}\big) \times \mathbb{P}(\mathbb{C}^n)$ is the graded ring

$$\mathcal{A}^*\left(\mathbb{P}\big(\mathbb{C}^{n\times n} \times \mathbb{C}\big) \times \mathbb{P}(\mathbb{C}^n)\right) = \mathbb{Z}[\omega_1, \omega_2]/(\omega_1^{n^2+1}, \omega_2^n),$$

where ω_1 and ω_2 denotes the rational equivalence classes of the inverse images of hyperplanes of $\mathbb{P}\big(\mathbb{C}^{n\times n} \times \mathbb{C}\big)$ and $\mathbb{P}(\mathbb{C}^n)$, under the projections $\mathbb{P}\big(\mathbb{C}^{n\times n} \times \mathbb{C}\big) \times \mathbb{P}(\mathbb{C}^n) \to \mathbb{P}\big(\mathbb{C}^{n\times n} \times \mathbb{C}\big)$ and $\mathbb{P}\big(\mathbb{C}^{n\times n} \times \mathbb{C}\big) \times \mathbb{P}(\mathbb{C}^n) \to \mathbb{P}(\mathbb{C}^n)$ respectively.

Given a codimension n algebraic subvariety $\mathcal{X} \subset \mathbb{P}\big(\mathbb{C}^{n\times n} \times \mathbb{C}\big) \times \mathbb{P}(\mathbb{C}^n)$, the class of $\mathcal{X}$ in the Chow ring is

$$[\mathcal{X}] = \sum_{i=0}^{n-1} \deg_{(n^2-1-i,i)}(\mathcal{X})\, \omega_1^{i+1}\omega_2^{n-1-i} \in \mathcal{A}^*\left(\mathbb{P}\big(\mathbb{C}^{n\times n} \times \mathbb{C}\big) \times \mathbb{P}(\mathbb{C}^n)\right).$$

Proof of Lemma 1.2.3. Let F_i, $(i = 1, \ldots, n)$, be the coordinate functions of F defined in (1.2.2). Since F_i is bilinear for each i, we have that the class of $\{F_i = 0\} \subset \mathbb{P}\big(\mathbb{C}^{n\times n} \times \mathbb{C}\big) \times \mathbb{P}(\mathbb{C}^n)$ is given by

$$[\{F_i = 0\}] = \omega_1 + \omega_2 \in \mathcal{A}^*\left(\mathbb{P}\big(\mathbb{C}^{n\times n} \times \mathbb{C}\big) \times \mathbb{P}(\mathbb{C}^n)\right), \quad (i = 1, \ldots, n).$$

Then, the class of $\mathcal{V}$ in the Chow ring is

$$[\mathcal{V}] = [\{F_1 = 0\} \cap \cdots \cap \{F_n = 0\}] = \prod_{i=1}^{n} [\{F_i = 0\}],$$

where the last equality follows from Bézout identity. Therefore, one gets

$$[\mathcal{V}] = (\omega_1 + \omega_2)^n \equiv \sum_{\ell=1}^{n} \binom{n}{\ell} \omega_1^{\ell} \omega_2^{n-\ell},$$

that is

$$\deg_{(n^2-1-i,i)}(\mathcal{V}) = \binom{n}{i+1}.$$

□

Proposition 1.2.1. *For all $A \in \mathbb{P}(\mathbb{C}^{n\times n}) - \Sigma$ we have $\#\pi^{-1}(A) = \deg_{(n^2-1,0)}(\mathcal{V}) = n$.*

Proof. Since $\mathbb{P}(\mathbb{C}^{n\times n}) - \Sigma$ is connected, the number of preimages under π is constant on it. From *Lemma 1.2.1* we get that the restriction $\pi_1|_{\mathcal{V}-\Sigma'} : \mathcal{V} - \Sigma' \to \mathbb{P}\big(\mathbb{C}^{n\times n} \times \mathbb{C}\big)$ is a bijective map onto its image $\pi_1(\mathcal{V} - \Sigma')$. Therefore, given $A \in \mathbb{P}(\mathbb{K}^{n\times n}) - \Sigma$, we have $\#\pi^{-1}(A) = \#p|_{\pi_1(\mathcal{V})}{}^{-1}(A)$, where p is the projection map given in (1.2.4). Moreover, from [Mumford, 1976, Corollary 5.6], we get that $\#p|_{\pi_1(\mathcal{V})}{}^{-1}(A) = \deg \pi_1(\mathcal{V})$, where deg is the degree of the projective algebraic subvariety $\pi_1(\mathcal{V}) \subset \mathbb{P}\big(\mathbb{C}^{n\times n} \times \mathbb{C}\big)$. Since $\dim \pi_1(\mathcal{V}) = \dim(\mathcal{V})$ and the fact that $\pi_1|_{\mathcal{V}-\Sigma'} : \mathcal{V} - \Sigma' \to \pi_1(\mathcal{V} - \Sigma')$ is bijective, we get that $\#(\Lambda \times \mathbb{P}(\mathbb{C}^n)) \cap \mathcal{V} = \#\Lambda \cap \pi_1(\mathcal{V})$, for a generic (n^2-1)-codimension plane $\Lambda \subset \mathbb{P}\big(\mathbb{C}^{n\times n} \times \mathbb{C}\big)$. Then, we obtain that $\deg \pi_1(\mathcal{V}) = \deg_{(n^2-1,0)}(\mathcal{V})$. □

Remark 1.2.4. From this proposition we get that the map $\pi|_{\mathcal{V}-\pi^{-1}(\Sigma)} : \mathcal{V} - \pi^{-1}(\Sigma) \to \mathbb{P}(\mathbb{C}^{n\times n}) - \Sigma$ is a n-fold covering map.

1.2.2.2 Unitary Invariance

Let $\mathbb{U}_n(\mathbb{K})$ stand for the unitary group when $\mathbb{K} = \mathbb{C}$ or the orthogonal group when $\mathbb{K} = \mathbb{R}$. The group $\mathbb{U}_n(\mathbb{K})$ acts on $\mathbb{P}(\mathbb{K}^n)$ in the natural way, and acts on $\mathbb{K}^{n\times n}$ by sending $A \mapsto UAU^{-1}$. Moreover if $(A, \lambda, v) \in \mathcal{V}$, then $(UAU^{-1}, \lambda, Uv) \in \mathcal{V}$.

Thus, $\mathcal{V}$ is invariant under the product action $\mathbb{U}_n(\mathbb{K}) \times \mathcal{V} \to \mathcal{V}$ given by

$$U \cdot (A, \lambda, v) \mapsto (UAU^{-1}, \lambda, Uv), \quad U \in \mathbb{U}_n(\mathbb{K}). \tag{1.2.6}$$

Remark 1.2.5. Note that the group $\mathbb{U}_n(\mathbb{K})$ preserves the Hermitian structure on $\mathcal{V}$, therefore $\mathbb{U}_n(\mathbb{K})$ acts by isometries on $\mathcal{V}$.

1.3 Condition Numbers

In this section we introduce the eigenvalue and eigenvector condition numbers, and we define the condition number for the eigenvalue problem. We will discuss the condition number theorem for this framework, which relates the condition number with the distance to ill-posedness. In the last part of this section we study the rate of change of condition numbers.

1.3.1 Eigenvalue and Eigenvector Condition Numbers

When (A, λ, v) belongs to $\mathcal{W}$, according to the implicit function theorem, π has an inverse defined in some neighborhood $\mathcal{U}_A \subset \mathbb{P}(\mathbb{K}^{n\times n})$ of A such that $\pi^{-1}(A) = (A, \lambda, v)$. This map $\mathscr{S} = \pi^{-1}|_{\mathcal{U}_A} : \mathcal{U}_A \to \mathcal{V}$ is called the *solution map*. It associates to any matrix $B \in \mathcal{U}_A$ the eigentriple (B, λ_B, v_B) close to (A, λ, v). Its derivative

$$D\mathscr{S}(A, \lambda, v) : T_A\mathbb{P}(\mathbb{K}^{n\times n}) \to T_{(A,\lambda,v)}\mathcal{V},$$

is called the *condition operator* at (A, λ, v).

If $(A, \lambda, v) \in \mathcal{W}$, the derivative $D\mathscr{S}(A, \lambda, v)$ associates to each $\dot{B} \in T_A\mathbb{P}(\mathbb{K}^{n\times n})$ a triple $(\dot{A}, \dot{\lambda}, \dot{v})$ satisfying (1.2.3). Moreover, equation (1.2.3) defines two linear maps,

$$D\mathscr{S}_\lambda(A, \lambda, v)\dot{B} = (\dot{A}, \dot{\lambda}) \quad \text{and} \quad D\mathscr{S}_v(A, \lambda, v)\dot{B} = \dot{v},$$

namely, the condition operators of the eigenvalue and eigenvector respectively.

Lemma 1.3.1. *Let $(A, \lambda, v) \in \mathcal{W}$. Then for $\dot{B} \in T_A\mathbb{P}(\mathbb{K}^{n\times n})$, one gets:*

(i)

$$D\mathscr{S}_\lambda(A,\lambda,v)\dot{B} = \big(\dot{B} - \dot{\lambda}\frac{\overline{\lambda}}{\|A\|_F^2}A, \dot{\lambda}\big), \quad \text{where } \dot{\lambda} = \frac{\langle \dot{B}v, u\rangle}{\left(1 + \frac{|\lambda|^2}{\|A\|_F^2}\right)\langle v, u\rangle};$$

(ii)

$$D\mathscr{S}_v(A,\lambda,v)\dot{B} = \Pi_{v^\perp}(\lambda I_n - A)|_{v^\perp}{}^{-1}\left(\Pi_{v^\perp}(\dot{B}v)\right),$$

where $u \in \mathbb{K}^n$ is a left eigenvector of A with eigenvalue λ: a non-zero vector satisfying $(\lambda I_n - A)^* u = 0$.

Proof. (i): Note that the relation between $\dot{B} \in A^\perp$ and $(\dot{A}, \dot{\lambda}) \in (A,\lambda)^\perp$ is given by

$$\dot{B} = \dot{A} + \frac{\dot{\lambda}\cdot\overline{\lambda}}{\|A\|_F^2}A. \tag{1.3.1}$$

Moreover, from (1.2.3) we get $\langle \dot{A}v, u\rangle = \dot{\lambda}\langle v, u\rangle$, i.e.

$$\dot{\lambda} = \frac{\langle \dot{A}v, u\rangle}{\langle v, u\rangle}. \tag{1.3.2}$$

From (1.3.1) and (1.3.2) follows

$$\dot{\lambda} = \frac{1}{1 + \frac{|\lambda|^2}{\|A\|_F^2}}\frac{\langle \dot{B}v, u\rangle}{\langle v, u\rangle}.$$

(ii): From (1.2.3) again one gets

$$\dot{v} = \Pi_{v^\perp}(\lambda I_n - A)|_{v^\perp}{}^{-1}\left(\Pi_{v^\perp}(\dot{A}v)\right).$$

Since $\Pi_{v^\perp}(\dot{B}v) = \Pi_{v^\perp}(\dot{A}v)$ by (1.3.1), the result follows. $\square$

Since $\mathbb{P}(\mathbb{K}^{n\times n})$ is equipped with the canonical Hermitian structure induced by the Frobenius Hermitian product on $\mathbb{K}^{n\times n}$, the condition numbers of the eigenvalue and eigenvector are given by

$$\mu_\lambda(A,\lambda,v) = \sup_{\substack{\dot{B}\in A^\perp \\ \|\dot{B}\|_F=\|A\|_F}} \|D\mathscr{S}_\lambda(A,\lambda,v)\dot{B}\|_{(A,\lambda)}$$
$$\mu_v(A,\lambda,v) = \sup_{\substack{\dot{B}\in A^\perp \\ \|\dot{B}\|_F=\|A\|_F}} \|D\mathscr{S}_v(A,\lambda,v)\dot{B}\|_v$$

Proposition 1.3.1. *Let $(A,\lambda,v)\in\mathcal{W}$. Then*

(i)

$$\mu_\lambda(A,\lambda,v) = \frac{1}{1+\frac{|\lambda|^2}{\|A\|_F^2}}\cdot\left[1+\frac{\|v\|^2\cdot\|u\|^2}{|\langle v,u\rangle|^2}\right]^{1/2};$$

(ii)

$$\mu_v(A,\lambda,v) = \|A\|_F\cdot\|\Pi_{v^\perp}(\lambda I_n - A)|_{v^\perp}{}^{-1}\|,$$

where $\|\cdot\|$ is the operator norm.

Remark 1.3.1. $\Pi_{v^\perp}(\lambda I_n - A)|_{v^\perp}$ is a linear map from the Hermitian complement of v in $\mathbb{K}^n$ into itself. Hence the operator norm of its inverse is independent of the representative of v.

Proof. (i): From *Lemma 1.3.1*,

$$\begin{aligned}\|D\mathscr{S}_\lambda(A,\lambda,v)\dot{B}\|_{(A,\lambda)}^2 &= \frac{\|\dot{B}\|_F^2+|\dot{\lambda}|^2\left(1+\frac{|\lambda|^2}{\|A\|_F^2}\right)}{\|A\|_F^2+|\lambda|^2} \\ &= \frac{\|\dot{B}\|_F^2+\left|\frac{\langle\dot{B}v,u\rangle}{\langle v,u\rangle}\right|^2\left(1+\frac{|\lambda|^2}{\|A\|_F^2}\right)^{-1}}{\|A\|_F^2+|\lambda|^2}.\end{aligned} \tag{1.3.3}$$

Then, the proof of (i) can be deduced from the following result:

$$\sup_{\substack{\dot{B}\in A^\perp \\ \|\dot{B}\|_F=\|A\|_F}} \left|\langle\dot{B}v,u\rangle\right| = \|A\|_F\cdot\sqrt{\|v\|^2\cdot\|u\|^2-\frac{|\lambda|^2}{\|A\|_F^2}\cdot|\langle v,u\rangle|^2}.$$

(The proof is left to the reader).

(ii): Since $Av = \lambda v$, we have $\Pi_{v^\perp}(\dot{B}v) = \Pi_{v^\perp}((\dot{B} + \alpha A)v)$, for any $\alpha \in \mathbb{K}$ and $\dot{B} \in A^\perp$. Then, from *Lemma 1.3.1* we get:

$$\begin{aligned}\mu_v(A, \lambda, v) &= \sup_{\substack{\dot{B}\in A^\perp \\ \|\dot{B}\|_F = \|A\|_F}} \left\| \Pi_{v^\perp}(\lambda I_n - A)|_{v^\perp}{}^{-1}\left(\Pi_{v^\perp}(\dot{B}v)\right)\right\|_v \\ &= \sup_{\substack{\dot{B}\in \mathbb{K}^{n\times n} \\ \|\dot{B}\|_F = 1}} \|A\|_F \cdot \left\| \Pi_{v^\perp}(\lambda I_n - A)|_{v^\perp}{}^{-1}\left(\Pi_{v^\perp}(\dot{B}v)\right)\right\|_v .\end{aligned}$$

Since $\{\Pi_{v^\perp}(\dot{B}v) : \ \dot{B} \in \mathbb{K}^{n\times n}, \|\dot{B}\|_F = 1\}$ fill the ball of radius $\|v\|$ in $v^\perp$, the result follows. □

Corollary 2. *μ_λ and μ_v are invariant under the action of $\mathbb{U}_n(\mathbb{K})$.*

Remark 1.3.2. Let $(A, \lambda, v) \in \mathcal{W}$. If $(\lambda I_n - A)^* v = 0$, that is, if v is also a left eigenvector of A with eigenvalue λ, then,

$$\mu_\lambda(A, \lambda, v) = \frac{\sqrt{2}}{1 + \frac{|\lambda|^2}{\|A\|_F^2}}.$$

In particular, this is the case when A is normal, i.e. $A^*A = AA^*$. On the other hand, μ_v happens to be more interesting since, roughly speaking, it measures how close to λ others eigenvalues are.

Lemma 1.3.2. *Let $A \in \mathbb{P}(\mathbb{K}^{n\times n})$ be a normal matrix. If $(A, \lambda, v) \in \mathcal{W}$ then*

$$\mu_v(A, \lambda, v) = \frac{\|A\|_F}{\min_i |\lambda - \lambda_i|},$$

where the minimum is taken for λ_i eigenvalue of A different from λ.

Proof. Since A is normal, by the unitary invariance of μ_v, we may assume that A is the diagonal matrix $\mathrm{Diag}(\lambda, \lambda_2, \ldots, \lambda_n)$, where λ, λ_i are the eigenvalues of A. Moreover, since $(A, \lambda, v) \in \mathcal{W}$, $\lambda \neq \lambda_i$ for $i = 2, \ldots n$. Then, the result follows from *Proposition 1.3.1.* □

1.3.2 Condition Number Revisited

The condition number of a computational problem is usually defined as the operator norm of the map giving the first order variation of the output in terms of the first order variation of the input. In our case the condition number should be the operator norm of the condition operator $D\mathscr{S}(A,\lambda,v)$ given in *Section 1.3*, i.e.

$$\|D\mathscr{S}(A,\lambda,v)\| := \sup_{\substack{\dot{B}\in A^{\perp} \\ \|\dot{B}\|_F=\|A\|_F}} \|D\mathscr{S}(A,\lambda,v)\dot{B}\|_{(A,\lambda,v)}.$$

Note that this quantity is bounded below by $\mu_v(A,\lambda,v)$ and above by $(\mu_\lambda(A,\lambda,v)^2 + \mu_v(A,\lambda,v)^2)^{1/2}$. However, in spite of this definition, we define the condition number of the eigenvalue problem in the following way.

Definition 2 (Condition Number)**.** The condition number of the eigenvalue problem is defined by

$$\mu(A,\lambda,v) := \max\{1, \mu_v(A,\lambda,v)\}. \tag{1.3.4}$$

In the next proposition we show that this definition and the usual one are essentially equivalent.

Proposition 1.3.2. *Let* $(A,\lambda,v) \in \mathcal{W}$*. Then*

$$\frac{1}{\sqrt{2}} \cdot \mu(A,\lambda,v) \leq \|D\mathscr{S}(A,\lambda,v)\| \leq 2 \cdot \mu(A,\lambda,v)$$

The proof follows from the next lemma.

Lemma 1.3.3. *Let* $(A,\lambda,v) \in \mathcal{W}$*. Then,*

(i) $\mu_v(A,\lambda,v) \geq 1/\sqrt{2}$*;*

(ii)

$$\mu_\lambda(A,\lambda,v) \leq \frac{1}{1+\frac{|\lambda|^2}{\|A\|_F^2}} \cdot (2+\mu_v(A,\lambda,v)^2)^{1/2}.$$

Proof. Fix a representative of $(A,\lambda,v) \in \mathcal{W}$ such that $\|v\| = 1$.
(i): One has,

$$\|\Pi_{v^\perp}(\lambda I_n - A)|_{v^\perp}\| \leq \|\Pi_{v^\perp}(A)|_{v^\perp}\| + |\lambda| \leq \sqrt{2}\|A\|_F,$$

that is, $\|\Pi_{v^\perp}(\lambda I_n - A)|_{v^\perp}\| \leq \sqrt{2}$. Therefore,

$$1 = \|\left(\Pi_{v^\perp}(\lambda I_n - A)|_{v^\perp}\right)^{-1} \Pi_{v^\perp}(\lambda I_n - A)|_{v^\perp}\| \leq \sqrt{2}\mu_v(A,\lambda,v).$$

(ii): Since the action of $\mathbb{U}_n(\mathbb{K})$ on $\mathbb{P}(\mathbb{K}^n)$ is transitive, we may assume that v is the first element of the canonical basis. Then A has the form $\begin{pmatrix} \lambda & w \\ 0 & \hat{A} \end{pmatrix}$, where $w \in \mathbb{K}^{1\times(n-1)}$ and $\hat{A} \in \mathbb{K}^{(n-1)\times(n-1)}$. Then $A - \lambda I_n = \begin{pmatrix} 0 & w \\ 0 & \hat{A} - \lambda I_{n-1} \end{pmatrix}$. Note that $u = (1, -(\hat{A} - \lambda I_{n-1})^{-*} w^*)^T$ is solution of $(A - \lambda I_n)^* u = 0$, i.e. u is a left eigenvector. Here, $\cdot^T$ and $\cdot^*$ denotes the transpose and conjugate transpose respectively. Then,

$$\begin{aligned}
\frac{|\langle v, u\rangle|}{\|v\| \cdot \|u\|} &= \frac{1}{\sqrt{1 + \|(\hat{A} - \lambda I_{n-1})^{-*} w^*\|^2}} \\
&\geq \frac{1}{\sqrt{1 + \|(\hat{A} - \lambda I_{n-1})^{-1}\|^2 \cdot \|w\|^2}} \\
&\geq \frac{1}{\sqrt{1 + \|(\hat{A} - \lambda I_{n-1})^{-1}\|^2 \cdot \|A\|_F^2}} = \frac{1}{\sqrt{1 + \mu_v(A, \lambda, v)^2}}.
\end{aligned}$$

The result now follows from *Proposition 1.3.1*. □

The next subsection is included for the sake of completeness but is not needed for the proof of our main results.

1.3.3 Condition Number Theorems

In this subsection we study the relation of $\mu_\lambda(A, \lambda, v)$, $\mu_v(A, \lambda, v)$ and $\mu(A, \lambda, v)$ with the distance of (A, λ, v) to Σ'. The main result in this subsection is that $\mu(A, \lambda, v)$ is bounded above by $\sin(d_{\mathbb{P}^2}(A, \lambda, v), \Sigma'))^{-1}$.

Let $(\mathbb{E}, \langle\cdot,\cdot\rangle)$ be a finite dimensional Hermitian vector space over $\mathbb{K}$. Given Λ a projective subset in $\mathbb{P}(\mathbb{E})$, we denote by $\hat{\Lambda} \subset \mathbb{E}$ its affine extension.

Lemma 1.3.4. *Given $x \in \mathbb{E}$, $x \neq 0$, we have*

$$\sin(d_{\mathbb{P}}(x, \Lambda)) = \frac{d_{\mathbb{E}}(A, \hat{\Lambda})}{\|x\|},$$

where $d_{\mathbb{E}}$ is the distance generated by $\langle\cdot,\cdot\rangle$.

Proof. The proof is straightforward. □

The next proposition is a version, adapted to this context, of known results given by Wilkinson [1972]) and Shub & Smale [1996] .

Recall that $\Sigma = \pi(\Sigma') \subset \mathbb{P}(\mathbb{K}^{n\times n})$.

Proposition 1.3.3. *Let* $(A, \lambda, v) \in \mathcal{W}$. *Then*

(i)

$$\mu_\lambda(A, \lambda, v) \leq \sqrt{\frac{2}{\sin(d_{\mathbb{P}}(A, \Sigma))^2} + 1};$$

(ii)

$$\mu_v(A, \lambda, v) = \frac{\|A\|_F}{d_F(A, \hat{\Sigma}_v + \lambda I_n)},$$

where $\Sigma_v = \{B \in \mathbb{P}(\mathbb{K}^{n\times n}) : (B, 0, v) \in \Sigma'\} \subset \Sigma$.

Proof. (i) Let $\hat{\Sigma} \subset \mathbb{K}^{n\times n}$ be the affine extension of Σ in $\mathbb{K}^{n\times n}$, and let u be a left eigenvector associated to A with eigenvalue λ. Wilkinson shows that:

$$\frac{\|v\| \cdot \|u\|}{|\langle v, u\rangle|} \leq \sqrt{2}\frac{\|A\|_F}{d_F(A, \hat{\Sigma})},$$

(cf. Demmel [1988], Wilkinson [1972]). Then, (i) follows from *Proposition 1.3.1* and *Lemma 1.3.4*.
(ii) In Shub & Smale [1996] it is proved that, for a fixed triple $(A, \lambda, v) \in \mathcal{V}$,

$$d_F(\lambda I_n - A, \hat{\Sigma}_v) = \frac{1}{\|\Pi_{v^\perp}(\lambda I_n - A)|_{v^\perp}{}^{-1}\|}.$$

Then, (ii) follows from *Lemma 1.3.1.* □

Corollary 3. *For* $(A, \lambda, v) \in \mathcal{W}$, *we get*

$$\mu(A, \lambda, v) \leq \frac{1}{\sin(d_{\mathbb{P}}(A, \Sigma))}.$$

Proof. Since $\hat{\Sigma}_v + \alpha I_n \subset \hat{\Sigma}$ for all $\alpha \in \mathbb{K}$, we conclude from *Lemma 1.3.4* that:

$$\mu_v(A, \lambda, v) \leq \frac{1}{\sin(d_{\mathbb{P}}(A, \Sigma))}.$$

Moveover, since the second member is greater than one, the proof follows. □

Proposition 1.3.4. *For $(A, \lambda, v) \in \mathcal{W}$, we get*

$$\mu(A, \lambda, v) \leq \frac{1}{\sin(d_{\mathbb{P}^2}((A, \lambda, v), \Sigma'))}.$$

Proof. Let $\Sigma''_v := \{(B, \eta) \in \mathbb{P}(\mathbb{K}^{n\times n} \times \mathbb{K}) : \ (B, \eta, v) \in \Sigma'\}$, and $\hat{\Sigma}''_v$ its affine extension in $\mathbb{K}^{n\times n} \times \mathbb{K}$. Note that

$$d_{\mathbb{K}^{n\times n}\times\mathbb{K}}((A, \lambda), \hat{\Sigma}''_v) = d_F(A - \lambda I_n, \hat{\Sigma}_v),$$

where Σ_v is defined in *Proposition 1.3.3*. Then, from *Proposition 1.3.3*, we get

$$d_{\mathbb{K}^{n\times n}\times\mathbb{K}}((A, \lambda), \hat{\Sigma}''_v) = \frac{\|A\|_F}{\mu_v(A, \lambda, v)}.$$

Since $\pi_1^{-1}(\Sigma''_v) \subset \Sigma'$, we get

$$d_{\mathbb{P}^2}((A, \lambda, v), \Sigma') \leq d_{\mathbb{P}^2}\big((A, \lambda, v), \pi_1^{-1}(\Sigma''_v)\big) = d_{\mathbb{P}}((A, \lambda), \Sigma''_v)).$$

Then, the result follows from the fact that $\sin(\cdot) \leq 1$. □

1.3.4 Condition Number Sensitivity

For the proof of *Theorem 2* we have to study the rate of change of the condition number μ defined in (1.3.4).

The main result of this subsection is the following.

Proposition 1.3.5. *Given $\varepsilon > 0$, there exist $C_\varepsilon > 0$ such that, if (A, λ, v), (A', λ', v') belongs to $\mathcal{W}$ and*

$$d_{\mathbb{P}^2}\big((A, \lambda, v), (A', \lambda', v')\big) < \frac{C_\varepsilon}{\mu(A, \lambda, v)}$$

then

$$\frac{\mu(A, \lambda, v)}{1 + \varepsilon} \leq \mu(A', \lambda', v') \leq (1 + \varepsilon)\mu(A, \lambda, v).$$

(One may choose $C_\varepsilon = \frac{\arctan\left(\frac{\varepsilon}{2\sqrt{2}+\sqrt{2}\alpha(1+\varepsilon)}\right)}{(1+\varepsilon)}$*, where* $\alpha := (1+\sqrt{5})2\sqrt{2}$*).*

Before proving *Proposition 1.3.5* we need some additional notation.

When $\mathbb{E}$ is a finite dimensional vector space over $\mathbb{K}$ equipped with the Hermitian inner product $\langle\cdot,\cdot\rangle$, we define

$$d_T(w,w') := \tan(d_{\mathbb{P}}(w,w')), \tag{1.3.5}$$

for all w, $w' \in \mathbb{P}(\mathbb{E})$. We have

$$d_T(w,w') = \|w-w'\|_w,$$

whenever w and w' satisfy $\langle w-w', w\rangle = 0$.

Note that $d_{\mathbb{P}}(\cdot,\cdot) \le d_T(\cdot,\cdot)$. Moreover, we have:

Lemma 1.3.5. *Let* w, $w' \in \mathbb{P}(\mathbb{E})$ *such that* $d_{\mathbb{P}}(w,w') \le \theta < \pi/2$*. Then*

$$d_{\mathbb{P}}(w,w') \le d_T(w,w') \le \frac{\tan(\theta)}{\theta}\cdot d_{\mathbb{P}}(w,w'), \quad \textit{for all } w,\ w' \in \mathbb{P}(\mathbb{E}).$$

Proof. This follows from elementary facts. □

Given $w \in \mathbb{K}^n$, $w \neq 0$, we define for any $B \in \mathbb{K}^{n\times n}$ the map

$$\hat{\Pi}_{w^\perp}B : \mathbb{K}^n \to \mathbb{K}^n, \quad \text{by} \quad \hat{\Pi}_{w^\perp}B := \tau \circ \Pi_{w^\perp}B,$$

where $\tau : w^\perp \to \mathbb{K}^n$ is the inclusion map. That is,

$$\hat{\Pi}_{w^\perp}Bz = Bz - \langle Bz, \frac{w}{\|w\|}\rangle\frac{w}{\|w\|}.$$

Since $\left(\hat{\Pi}_{v^\perp}(\lambda I_n - A)\right)v = 0$ for all $(A,\lambda,v) \in \mathcal{W}$, then we have

$$\begin{aligned}\mu_v(A,\lambda,v) &= \|A\|_F \cdot \left\|\Pi_{v^\perp}(\lambda I_n - A)|_{v^\perp}{}^{-1}\right\| \\ &= \|A\|_F \cdot \left\|\left(\hat{\Pi}_{v^\perp}(\lambda I_n - A)\right)^\dagger\right\|,\end{aligned}$$

where $\dagger$ is the Moore-Penrose inverse.

Lemma 1.3.6. *Let* v, $w \in \mathbb{P}(\mathbb{K}^n)$ *and* $B \in \mathbb{K}^{n\times n}$. *Then*

$$\left\|\hat{\Pi}_{v^\perp}B - \hat{\Pi}_{w^\perp}B\right\| \leq 2\|B\| \cdot d_T(v,w).$$

Proof. Take representatives of v and w such that $\|v\| = 1$ and $\langle v-w, v\rangle = 0$. Let $u \in \mathbb{K}^n$, then

$$\begin{aligned}
\left\|\left(\hat{\Pi}_{v^\perp}B - \hat{\Pi}_{w^\perp}B\right)u\right\| &= \left\|Bu - \langle Bu, v\rangle v - \left(Bu - \langle Bu, \frac{w}{\|w\|}\rangle\frac{w}{\|w\|}\right)\right\| \\
&= \left\|\langle Bu, \frac{w}{\|w\|}\rangle\frac{w}{\|w\|} - \langle Bu, v\rangle v\right\| \\
&\leq \left\|\langle Bu, \frac{w}{\|w\|} - v\rangle\frac{w}{\|w\|} + \langle Bu, v\rangle\left(\frac{w}{\|w\|} - v\right)\right\| \\
&\leq 2\|Bu\| \cdot \left\|\frac{w}{\|w\|} - v\right\| \leq 2\|Bu\| \cdot d_T(v,w).
\end{aligned}$$

□

Let d_{T^2} be the product function defined over $\mathbb{P}\big(\mathbb{K}^{n\times n} \times \mathbb{K}\big) \times \mathbb{P}(\mathbb{K}^n)$ by

$$d_{T^2}((A,\lambda,v),(A',\lambda',v')) := \big(d_T((A,\lambda),(A',\lambda'))^2 + d_T(v,v')^2\big)^{1/2}$$

Proposition 1.3.6. *Let* $\alpha := (1+\sqrt{5})2\sqrt{2}$.
Let $(A,\lambda,v), (A',\lambda',v') \in \mathcal{W}$ *such that*

$$d_{T^2}\big((A,\lambda,v),(A',\lambda',v')\big) < \frac{1}{\alpha \cdot \mu_v(A,\lambda,v)}.$$

Then, the following inequality holds:

$$\mu_v(A',\lambda',v') \leq \frac{(1+\sqrt{2}d_{T^2}\big((A,\lambda,v),(A',\lambda',v')\big)) \cdot \mu_v(A,\lambda,v)}{1 - \alpha \cdot \mu_v(A,\lambda,v) \cdot d_{T^2}\big((A,\lambda,v),(A',\lambda',v')\big)}.$$

Proof. Consider representatives of (A,λ,v) and (A',λ',v') such that: $\|A\|_F = \|v\| = 1$, $(A,\lambda) - (A',\lambda')$ perpendicular to (A,λ) in $\mathbb{K}^{n\times n} \times \mathbb{K}$, and $v - v'$ perpendicular to v in $\mathbb{K}^n$.
Notation: for short, let $A_\lambda := (\lambda I_n - A)$ and $A'_{\lambda'} := (\lambda' I_n - A')$.

By Wedin's Theorem (see Stewart & Sun [1990], Theorem 3.9) we have

$$\left\|\left(\hat{\Pi}_{v^\perp}A_\lambda\right)^\dagger - \left(\hat{\Pi}_{v'^\perp}A'_{\lambda'}\right)^\dagger\right\| \leq \frac{1+\sqrt{5}}{2}\cdot\left\|\left(\hat{\Pi}_{v^\perp}A_\lambda\right)^\dagger\right\|\cdot\left\|\left(\hat{\Pi}_{v'^\perp}A'_{\lambda'}\right)^\dagger\right\|\cdot\left\|\hat{\Pi}_{v^\perp}A_\lambda - \hat{\Pi}_{v'^\perp}A'_{\lambda'}\right\|.$$

Since $\left|\left\|\left(\hat{\Pi}_{v^\perp}A_\lambda\right)^\dagger\right\| - \left\|\left(\hat{\Pi}_{v'^\perp}A'_{\lambda'}\right)^\dagger\right\|\right| \leq \left\|\left(\hat{\Pi}_{v^\perp}A_\lambda\right)^\dagger - \left(\hat{\Pi}_{v'^\perp}A'_{\lambda'}\right)^\dagger\right\|$, then,

$$\left\|\left(\hat{\Pi}_{v'^\perp}A'_{\lambda'}\right)^\dagger\right\| \leq \frac{\left\|\left(\hat{\Pi}_{v^\perp}A_\lambda\right)^\dagger\right\|}{1-\frac{1+\sqrt{5}}{2}\cdot\left\|\left(\hat{\Pi}_{v^\perp}A_\lambda\right)^\dagger\right\|\cdot\left\|\hat{\Pi}_{v^\perp}A_\lambda - \hat{\Pi}_{v'^\perp}A'_{\lambda'}\right\|}.$$

Note that

$$\begin{aligned}\left\|\hat{\Pi}_{v^\perp}A_\lambda - \hat{\Pi}_{v'^\perp}A'_{\lambda'}\right\| &\leq \left\|\hat{\Pi}_{v^\perp}A_\lambda - \hat{\Pi}_{v'^\perp}A_\lambda\right\| + \left\|\hat{\Pi}_{v'^\perp}A_\lambda - \hat{\Pi}_{v'^\perp}A'_{\lambda'}\right\| \\ &\leq 2\cdot\|A_\lambda\|\cdot d_T(v,v') + \|A_\lambda - A'_{\lambda'}\|,\end{aligned}$$

where the second inequality follows from *Lemma 1.3.6*. Moreover, taking into account that $(A,\lambda,v)\in\mathcal{W}$ and the choice of elected representatives, we get

$$\begin{aligned}\|A_\lambda - A'_{\lambda'}\| &\leq \|A-A'\| + |\lambda-\lambda'| \\ &\leq \sqrt{2}\cdot d_T((A,\lambda),(A',\lambda'))\cdot\sqrt{\|A\|_F^2+|\lambda|^2} \\ &\leq 2\cdot d_T((A,\lambda),(A',\lambda')),\end{aligned}$$

and hence

$$\left\|\hat{\Pi}_{v^\perp}A_\lambda - \hat{\Pi}_{v'^\perp}A'_{\lambda'}\right\| \leq 4\cdot d_T(v,v') + 2\cdot d_T((A,\lambda),(A',\lambda')).$$

Then we conclude

$$\left\| \left(\hat{\Pi}_{v'^{\perp}} A'_{\lambda'} \right)^{\dagger} \right\| \leq$$

$$\frac{\left\| \left(\hat{\Pi}_{v^{\perp}} A_{\lambda} \right)^{\dagger} \right\|}{1-(1+\sqrt{5})2\sqrt{2}\cdot \left\| \left(\hat{\Pi}_{v^{\perp}} A_{\lambda} \right)^{\dagger} \right\| \cdot d_{T^2}((A,\lambda,v),(A',\lambda',v'))}.$$

The proposition follows from the following fact: $\|A'\|_F \leq 1 + \|A-A'\|_F \leq 1+\sqrt{2}d_T((A,\lambda),(A',\lambda'))$. □

Proposition 1.3.7. *Given $\varepsilon > 0$, there exist $c_\varepsilon > 0$ such that, if (A,λ,v), $(A',\lambda',v') \in \mathcal{W}$ and*

$$d_{T^2}\big((A,\lambda,v),(A',\lambda',v')\big) < \frac{c_\varepsilon}{\mu(A,\lambda,v)},$$

then,

$$\mu(A',\lambda',v') \leq (1+\varepsilon)\mu(A,\lambda,v).$$

(One may choose $c_\varepsilon = \frac{\varepsilon}{2\sqrt{2}+\sqrt{2}\alpha(1+\varepsilon)}$, where $\alpha = (1+\sqrt{5})2\sqrt{2}$.)

Proof. It is enough to prove the assertion for μ_v instead of μ.
Recall from *Lemma 1.3.3* that μ_v is bounded below by $1/\sqrt{2}$. Hence,

$$d_{T^2}\big((A,\lambda,v),(A',\lambda',v')\big) < \frac{c}{\mu(A,\lambda,v)},$$

implies

$$d_{T^2}\big((A,\lambda,v),(A',\lambda',v')\big) < \frac{\sqrt{2}\,c}{\mu_v(A,\lambda,v)}.$$

From *Proposition 1.3.6*, if c is such that $\sqrt{2}c < 1/\alpha$ and

$$\frac{1+2\sqrt{2}c}{1-\sqrt{2}\alpha c} < 1+\varepsilon,$$

we get the result.
One may choose $c_\varepsilon = \dfrac{\varepsilon}{2\sqrt{2}+\sqrt{2}\alpha(1+\varepsilon)}$.

□

Corollary 4. *Given $\varepsilon > 0$, there exist $c'_\varepsilon > 0$ such that, if (A, λ, v), $(A', \lambda', v') \in \mathcal{W}$ and*

$$d_{\mathbb{P}^2}\big((A,\lambda,v),(A',\lambda',v')\big) < \frac{c'_\varepsilon}{\mu(A,\lambda,v)},$$

then,

$$\mu(A',\lambda',v') \leq (1+\varepsilon)\mu(A,\lambda,v).$$

(One may choose $c'_\varepsilon = \arctan\left(\frac{\varepsilon}{2\sqrt{2}+\sqrt{2}\alpha(1+\varepsilon)}\right)$ where $\alpha := (1+\sqrt{5})2\sqrt{2}$.)

Proof. By *Lemma 1.3.4*, if

$$d_{\mathbb{P}^2}\big((A,\lambda,v),(A',\lambda',v')\big) < \frac{c'}{\mu(A,\lambda,v)},$$

then

$$\begin{aligned} d_{T^2}\big((A,\lambda,v),(A',\lambda',v')\big) &\leq \frac{\tan(c')}{c'} \cdot d_{\mathbb{P}^2}\big((A,\lambda,v),(A',\lambda',v')\big) \\ &< \frac{\tan(c')}{\mu(A,\lambda,v)}, \end{aligned}$$

proving the lemma. $\square$

Proof of Proposition 1.3.5. From *Corollary 4*, there exist $c' > 0$ such that, if (A, λ, v), $(A', \lambda', v') \in \mathcal{W}$ are such that

$$d_{\mathbb{P}^2}\big((A,\lambda,v),(A',\lambda',v')\big) \cdot \mu(A,\lambda,v) < c',$$

then

$$\mu(A',\lambda',v') \leq (1+\varepsilon)\mu(A,\lambda,v).$$

It is enough to take c' such that $c' \leq \arctan\left(\frac{\varepsilon}{2\sqrt{2}+\sqrt{2}\alpha(1+\varepsilon)}\right)$. In this case we have

$$d_{\mathbb{P}^2}\big((A,\lambda,v),(A',\lambda',v')\big) \cdot \mu(A',\lambda',v') < c'(1+\varepsilon).$$

Then, by the same argument, if $c'(1+\varepsilon) \leq \arctan\left(\frac{\varepsilon}{2\sqrt{2}+\sqrt{2}\alpha(1+\varepsilon)}\right)$ we have the other inequality. $\square$

1.4 Newton's Method

In this section we start describing the Newton method defined in the *Introduction*. The main goal of this section is to prove *Theorem 1*.

1.4.1 Introduction

Let us recall the definition of the Newton map on $\mathbb{P}\big(\mathbb{K}^{n\times n}\times\mathbb{K}\big)\times\mathbb{P}(\mathbb{K}^n)$. We define

$$N(A,\lambda,v) := \big(A, N_A(\lambda,v)\big),$$

where N_A is the Newton map, given in (1.1.1), associated to the evaluation map $F_A(\lambda,v)=(\lambda I_n - A)v$, for a fixed (non-zero) matrix A and $(\lambda,v)\in\mathbb{K}\times\mathbb{K}^n$.

Note that N_A has the simple matrix expression

$$N_A\begin{pmatrix}\lambda\\ v\end{pmatrix}=\begin{pmatrix}\lambda\\ v\end{pmatrix}-\begin{pmatrix}v & \lambda I_n - A\\ 0 & v^*\end{pmatrix}^{-1}\begin{pmatrix}(\lambda I_n - A)v\\ 0\end{pmatrix}.$$

To compute the Newton map we have to solve for $(\dot\lambda,\dot v)\in\mathbb{K}\times\mathbb{K}^n$ the following linear system:

$$\begin{aligned}\dot\lambda v + (\lambda I_n - A)\dot v &= (\lambda I_n - A)v,\\ \langle \dot v, v\rangle &= 0.\end{aligned}$$

Then one gets:

Lemma 1.4.1. *If $\Pi_{v^\perp}(\lambda I_n - A)|_{v^\perp}$ is invertible, then the Newton iteration is given by*

$$N(A,\lambda,v)=(A,\lambda-\dot\lambda, v-\dot v),$$

where

$$\begin{aligned}\dot v &= \big(\Pi_{v^\perp}(\lambda I_n - A)\big|_{v^\perp}\big)^{-1}\Pi_{v^\perp}(\lambda I_n - A)v,\\ \dot\lambda &= \frac{\langle(\lambda I_n - A)(v-\dot v), v\rangle}{\langle v, v\rangle}.\end{aligned}$$

From *Lemma 1.4.1*, we conclude that N is a well-defined map on the product space $\mathbb{P}\big(\mathbb{K}^{n\times n}\times\mathbb{K}\big)\times\mathbb{P}(\mathbb{K}^n)$. Moreover, for a fixed matrix $A\in\mathbb{K}^{n\times n}$, $A\neq 0_n$, we conclude also that the map N_A is well-defined on $\mathbb{K}\times\mathbb{P}(\mathbb{K}^n)$.

1.4.2 γ-Theorem

In order to prove *Theorem 1* and *Theorem 2* we need to obtain a version of the γ-Theorem for the Newton map $N_A : \mathbb{K} \times \mathbb{P}(\mathbb{K}^n) \to \mathbb{K} \times \mathbb{P}(\mathbb{K}^n)$.

Proposition 1.4.1. *Let $0 < u \leq 1/(2\sqrt{2})$.*
Let $(A, \lambda, v) \in \mathcal{W}$ such that $\|A\|_F = 1$, and let $(\lambda_0, v_0) \in \mathbb{K} \times \mathbb{P}(\mathbb{K}^n)$.
If

$$(|\lambda_0 - \lambda|^2 + d_{\mathbb{P}}(v_0, v)^2)^{1/2} < \frac{u}{\mu(A, \lambda, v)},$$

then, the Newton sequence $(\lambda_k, v_k) := N_A^k(\lambda_0, v_0)$ satisfies

$$(|\lambda_k - \lambda|^2 + d_{\mathbb{P}}(v_k, v)^2)^{1/2} \leq \left(\frac{2\tan(u)}{1 - \sqrt{2}\,u}\right) \cdot \left(\frac{1}{2}\right)^{2^k - 1} \cdot (|\lambda_0 - \lambda|^2 + d_{\mathbb{P}}(v_0, v)^2)^{1/2},$$

for all $k > 0$.

This proposition will be the main tool to prove *Theorem 1* and also *Theorem 2*. It is a version -for the Newton map N_A- of a fairly known theorem in the literature, namely, the γ-Theorem, which gives the size of the basin of attraction of Newton's method. In our case, for the Newton map N_A reads:

Theorem 3. *There is a universal constant $c_0 > 0$ with the following property. Let $(A, \lambda, v) \in \mathcal{W}$ such that $\|A\|_F = 1$, and $(\lambda_0, v_0) \in \mathbb{K} \times \mathbb{P}(\mathbb{K}^n)$. If*

$$(|\lambda_0 - \lambda|^2 + d_{\mathbb{P}}(v_0, v)^2)^{1/2} < \frac{c_0}{\mu(A, \lambda, v)},$$

then, the sequence $(\lambda_k, v_k) = N_A^k(\lambda_0, v_0)$ converges immediately quadratically to (λ, v) with respect to the canonical distance in $\mathbb{K} \times \mathbb{P}(\mathbb{K}^n)$.
(One may choose $c_0 = 0.288$).

Since we do not find an appropriate place to refer to this version, we include a proof of the *Proposition 1.4.1* in the *Appendix*. Note that the proof of *Theorem 3* follows directly from *Proposition 1.4.1* picking u such that: $0 < u \leq 1/(2\sqrt{2})$ and $2\tan(u)/(1 - \sqrt{2}u) \leq 1$.

1.4.3 Proof of Theorem 1

Preliminaries

Lemma 1.4.2. *Fix a representative of* $(A, \lambda, v) \in \mathcal{V}$ *such that* $\|A\|_F = 1$ *and* $\|v\| = 1$. *Let* $(\lambda', v') \in \mathbb{K} \times \mathbb{P}(\mathbb{K}^n)$.

1. *If* $|\lambda - \lambda'| \leq c < \sqrt{2}$, *then,*

$$d_{\mathbb{P}^2}\big((A,\lambda,v),(A,\lambda',v')\big) \leq \beta_c \cdot \left(|\lambda-\lambda'|^2 + d_{\mathbb{P}}(v,v')^2\right)^{1/2},$$

where $\beta_c = (1 - c^2/2)^{-1/2}$.

2. *If* $d_{\mathbb{P}^2}\big((A,\lambda,v),(A,\lambda',v')\big) < \theta < \pi/4$, *then,*

$$(|\lambda-\lambda'|^2 + d_T(v,v')^2)^{1/2} \leq R_\theta \cdot d_{\mathbb{P}^2}((A,\lambda.v),(A,\lambda',v')),$$

where $R_\theta = [\sqrt{2}/\cos(\theta+\pi/4)^3]^{1/2}$.

The proof of *Lemma 1.4.2* is included in the *Appendix.*

Let θ_0 such that $R_{\theta_0}\,\theta_0 = 1/(2\sqrt{2})$, where R_θ is given in *Lemma 1.4.2* ($\theta_0 \approx 0.1389$).

Proposition 1.4.2. *Let* $0 < u \leq \theta_0$.
Let (A,λ,v), $(A,\lambda_0,v_0) \in \mathbb{P}\big(\mathbb{K}^{n\times n}\times\mathbb{K}\big)\times\mathbb{P}(\mathbb{K}^n)$. *If* $(A,\lambda,v) \in \mathcal{W}$ *and*

$$d_{\mathbb{P}^2}\big((A,\lambda,v),(A,\lambda_0,v_0)\big) < \frac{u}{\mu(A,\lambda,v)},$$

then

$$d_{\mathbb{P}^2}\big(N^k(A,\lambda_0,v_0),(A,\lambda,v)\big) \leq$$
$$\leq R_u\,\beta_{u\,R_u}\left(\frac{2\tan(u\,R_u)}{1-\sqrt{2}\,u\,R_u}\right)\cdot\left(\frac{1}{2}\right)^{2^k-1} d_{\mathbb{P}^2}\big((A,\lambda,v),(A,\lambda_0,v_0)\big),$$

for all $k > 0$, *where* $\delta(u) := u/(1-u)$.

Proof. With out loss of generality we may assume $\|A\|_F = 1$ and $\|v\| = 1$.

By *Lemma 1.4.2* we get

$$(|\lambda_0-\lambda|^2+d_{\mathbb{P}}(v_0,v)^2)^{1/2} \le \frac{uR_u}{\mu(A,\lambda,v)}. \tag{1.4.1}$$

Since $u \le \theta_0$, we have $uR_u \le 1/(2\sqrt{2})$, and then from *Proposition 1.4.1* and *Proposition 1.6.2*, we get

$$(|\lambda_k-\lambda|^2+d_{\mathbb{P}}(v_k,v)^2)^{1/2} \le \\ \left(\frac{2\tan(u\,R_u)}{1-\sqrt{2}\,u\,R_u}\right)\cdot\left(\frac{1}{2}\right)^{2^k-1}\cdot(|\lambda_0-\lambda|^2+d_{\mathbb{P}}(v_0,v)^2)^{1/2}, \tag{1.4.2}$$

for all $k > 0$, where $(\lambda_k, v_k) := N_A^k(\lambda_0, v_0)$. Moreover, since $(|\lambda_0-\lambda|^2 + d_{\mathbb{P}}(v_0,v)^2)^{1/2} < u\,R_u$, we deduce from *Lemma 1.4.2* that

$$\begin{aligned} &d_{\mathbb{P}^2}(N^k(A,\lambda_0,v_0),(A,\lambda,v)) \le \\ &\quad\le \beta_{u\,R_u}\left(\frac{2\tan(u\,R_u)}{1-\sqrt{2}\,u\,R_u}\right)\cdot\left(\frac{1}{2}\right)^{2^k-1}\cdot(|\lambda_0-\lambda|^2+d_{\mathbb{P}}(v_0,v)^2)^{1/2} \\ &\quad\le R_u\,\beta_{u\,R_u}\left(\frac{2\tan(u\,R_u)}{1-\sqrt{2}\,u\,R_u}\right)\cdot\left(\frac{1}{2}\right)^{2^k-1}\cdot d_{\mathbb{P}^2}((A,\lambda_0,v_0),(A,\lambda,v)). \end{aligned}$$

(Note that $u \le \theta_0 < \frac{\pi}{4}$.) □

Proof of *Theorem 1*

Proof of Theorem 1. From *Proposition 1.4.2*, proof of *Theorem 1* follows picking $u_0 > 0$ such that $u_0 \le \theta_0$ and $R_{u_0}\,\beta_{u_0\,R_{u_0}}\left(\frac{2\tan(u_0\,R_{u_0})}{1-\sqrt{2}\,u_0\,R_{u_0}}\right) \le 1$. One may choose $u_0 = 0.0739$. □

1.5 Proof of the Main Theorem

1.5.1 Complexity Bound

In the introduction we defined the condition length of an absolutely continuous path $\Gamma : [a, b] \to \mathcal{W}$ as

$$\ell_\mu(\Gamma) = \int_a^b \|\dot{\Gamma}(t)\|_{\Gamma(t)}\, \mu(\Gamma(t))\, dt.$$

The next proposition is useful for our *Main Theorem 2*.

Proposition 1.5.1. *Given $\varepsilon > 0$, $C_\varepsilon > 0$ as in Proposition 1.3.5, and $\Gamma : [a, b] \to \mathcal{W}$ an absolutely continuous path with $\ell_\mu(\Gamma) < \infty$, define the real sequence $\{s_k\}_{k=0,\dots,K}$ in $[a, b]$ such that:*
•$s_0 = a$;
•s_k such that $\mu(\Gamma(s_{k-1})) \int_{s_{k-1}}^{s_k} \|\dot{\Gamma}(t)\|_{\Gamma(t)} dt = C_\varepsilon$,
whenever $\mu(\Gamma(s_{k-1})) \int_{s_{k-1}}^{b} \|\dot{\Gamma}(t)\|_{\Gamma(t)} dt > C_\varepsilon$;
• else define $s_k = s_K = b$.

Then,

$$K \leq \frac{1+\varepsilon}{C_\varepsilon}\, \ell_\mu(\Gamma) + 1.$$

Proof. Given $s \in [s_{k-1}, s_k]$, $d_{\mathbb{P}^2}(\Gamma(s_{k-1}), \Gamma(s)) \leq \int_{s_{k-1}}^{s_k} \|\dot{\Gamma}(t)\|_{\Gamma(t)} dt \leq \mu(\Gamma(s_{k-1}))^{-1}\, C_\varepsilon$. By the first inequality in *Proposition 1.3.5*, we get

$$\int_{s_{k-1}}^{s_k} \|\dot{\Gamma}(t)\|_{\Gamma(t)} \mu(\Gamma(t))\, dt \geq \frac{\mu(\Gamma(s_{k-1}))}{1+\varepsilon} \int_{s_{k-1}}^{s_k} \|\dot{\Gamma}(t)\|_{\Gamma(t)} dt = \frac{C_\varepsilon}{1+\varepsilon},$$

whenever $k < K$. Since $\ell_\mu(\Gamma) < \infty$, $K < \infty$, and adding, yields

$$\ell_\mu(\Gamma) \geq (K-1) \frac{C_\varepsilon}{1+\varepsilon}.$$

□

1.5.2 Proof of the Main Theorem 2

Proof of Theorem 2: Let $A(t)$, $a \leq t \leq b$, be a representative path of the projection $\pi(\Gamma) \subset \mathbb{P}(\mathbb{K}^{n\times n})$ such that $\|A(t)\|_F = 1$ for $a \leq t \leq b$.

Given a mesh $a = t_0 < t_1 < \ldots < t_K = b$, for $k = 0, 1, \ldots, K$, let $\hat{\Gamma}(t_k) = (A(t_k), \lambda_k, v_k) \in \mathbb{K}^{n\times n} \times \mathbb{K} \times \mathbb{P}(\mathbb{K}^n)$, be the sequence described in *Section 1.1.3.* Then, if $\Gamma(t_k)$, $\hat{\Gamma}(t_k)$, $\Gamma(t_{k+1})$ are such that,

$$d_{\mathbb{P}^2}(\Gamma(t_k), \Gamma(t_{k+1})) < \frac{C_\varepsilon}{\mu(\Gamma(t_k))}, \quad \text{and} \quad d_{\mathbb{P}^2}(\Gamma(t_k), \hat{\Gamma}(t_k)) < \frac{C_\varepsilon}{\mu(\Gamma(t_k))},$$

then,

$$\begin{aligned} d_{\mathbb{P}^2}(\Gamma(t_{k+1}), (A(t_{k+1}), \lambda_k, v_k)) \leq & \\ \leq d_{\mathbb{P}^2}(\Gamma(t_{k+1}), \Gamma(t_k)) + d_{\mathbb{P}^2}(\Gamma(t_k), \hat{\Gamma}(t_k)) + & \\ d_{\mathbb{P}^2}(\hat{\Gamma}(t_k), (A(t_{k+1}), \lambda_k, v_k)) & \\ \leq \frac{2C_\varepsilon}{\mu(\Gamma(t_k))} + d_{\mathbb{P}^2}(\hat{\Gamma}(t_k), (A(t_{k+1}), \lambda_k, v_k)). & \end{aligned}$$

Note that

$$d_{\mathbb{P}^2}(\hat{\Gamma}(t_k), (A(t_{k+1}), \lambda_k, v_k)) = d_{\mathbb{P}}((A(t_k), \lambda_k), (A(t_{k+1}), \lambda_k)).$$

Since $\|A(t)\|_F = 1$, $a \leq t \leq b$, then, abusing notation, we get

$$d_{\mathbb{P}}((A(t_k), \lambda_k), (A(t_{k+1}), \lambda_k)) \leq d_{\mathbb{P}}(A(t_k), A(t_{k+1})),$$

where the inequality follows from a direct application of the law of cosines. Moreover,

$$\begin{aligned} d_{\mathbb{P}}(A(t_k), A(t_{k+1})) &\leq \int_{t_k}^{t_{k+1}} \|\dot{A}(t)\|_{A(t)}\, dt \\ &\leq \sqrt{2} \int_{t_k}^{t_{k+1}} \|D\mathscr{S}_\lambda(\Gamma(t))\dot{A}(t)\|_{(A(t),\lambda(t))}\, dt \\ &\leq \sqrt{2} \int_{t_k}^{t_{k+1}} \|\dot{\Gamma}(t)\|_{\Gamma(t)}\, dt, \end{aligned}$$

where the second inequality follows from the trivial lower bound which one may obtain from (1.3.3).

Assuming $\int_{t_k}^{t_{k+1}} \|\dot{\Gamma}(t)\|_{\Gamma(t)}\, dt \leq C_\varepsilon/\mu(\Gamma(t_k))$ we conclude

$$d_{\mathbb{P}^2}(\Gamma(t_{k+1}), (A(t_{k+1}), \lambda_k, v_k)) \leq \frac{(2+\sqrt{2})C_\varepsilon}{\mu(\Gamma(t_k))}.$$

Since $d_{\mathbb{P}^2}(\Gamma(t_k), \Gamma(t_{k+1})) < \frac{C_\varepsilon}{\mu(\Gamma(t_k))}$, from *Proposition 1.3.5* we get

$$d_{\mathbb{P}^2}(\Gamma(t_{k+1}), (A(t_{k+1}), \lambda_k, v_k)) \leq \frac{(1+\varepsilon)(2+\sqrt{2})C_\varepsilon}{\mu(\Gamma(t_{k+1}))}.$$

From *Proposition 1.4.2*, if $u := (1+\varepsilon)C_\varepsilon(2+\sqrt{2}) \leq \theta_0$, then

$$\begin{aligned}
&d_{\mathbb{P}^2}(N(A(t_{k+1}), \lambda_k, v_k)), \Gamma(t_{k+1})) \leq \\
&\leq \; R_u\, \beta_{u\,R_u} \left(\frac{2\tan(u\,R_u)}{1-\sqrt{2}\,u\,R_u} \right) \frac{1}{2} \cdot d_{\mathbb{P}^2}(\Gamma(t_{k+1}), (A(t_{k+1}), \lambda_k, v_k)) \\
&\leq \; \frac{R_u\, \beta_{u\,R_u} \left(\frac{2\tan(u\,R_u)}{1-\sqrt{2}\,u\,R_u} \right) \frac{1}{2}\, u}{\mu(\Gamma(t_{k+1}))}.
\end{aligned}$$

Then, if ε is small enough such that $u \leq \theta_0$ and $R_u\, \beta_{u\,R_u} \left(\frac{2\tan(u\,R_u)}{1-\sqrt{2}\,u\,R_u} \right) \frac{1}{2}\, u < C_\varepsilon$, we get that $\hat{\Gamma}(t_{k+1})$ is an approximate solution of the eigenvalue problem $\Gamma(t_{k+1})$. Then, the proof of *Theorem 2* can be deduced applying *Proposition 1.5.1* to the ε selected before. $\square$

Remark 1.5.1. One can take $\varepsilon = 0.2448$. Then, $C_\varepsilon \approx 0.010383$, and one can choose $C = 120$.

1.6 Appendix

This section is divided in two parts. In the first one we include a proof of *Proposition 1.4.1.* In the second part we prove *Lemma 1.4.2.*

Proof of Proposition 1.4.1

Throughout this subsection, when ever we fix a representative of $(A, \lambda, v) \in \mathcal{W}$ such that $\|A\|_F = 1$ and $\|v\| = 1$, we will consider the canonical Hermitian structure on $\mathbb{K} \times \mathbb{K}^n$.

Preliminaries and Technical Lemmas

Lemma 1.6.1. *Let* $(A, \lambda, v) \in \mathcal{W}$ *and* $v' \in \mathbb{P}(\mathbb{K}^n)$ *such that* $d_{\mathbb{P}}(v, v') < \pi/2$.
1) Let $\Pi_{v^\perp}|_{v'^\perp} : v'^\perp \to v^\perp$ *be the restriction of the orthogonal projection* $\Pi_{v^\perp}$ *of* $\mathbb{K}^n$ *onto* $v'^\perp$. *Then,*

$$\| (\Pi_{v^\perp}|_{v'^\perp})^{-1} \| = \frac{1}{\cos(d_{\mathbb{P}}(v, v'))}.$$

2) Pick a representative of $(A, \lambda, v) \in \mathcal{W}$ *such that* $\|A\|_F = 1$ *and* $\|v\| = 1$. *Then,*

(i)

$$\| \left(DF_A(\lambda, v)|_{\mathbb{K}\times v'^\perp}\right)^{-1} \cdot DF_A(\lambda, v)|_{\mathbb{K}\times v^\perp} \| = \frac{1}{\cos(d_{\mathbb{P}}(v, v'))};$$

(ii)

$$\| \left(DF_A(\lambda, v)|_{\mathbb{K}\times v'^\perp}\right)^{-1} \| \leq \frac{\| (DF_A(\lambda, v)|_{\mathbb{K}\times v^\perp})^{-1} \|}{\cos(d_{\mathbb{P}}(v, v'))}.$$

Remark 1.6.1. In part 2) and 3) of the preceding lemma, we consider the spaces $\mathbb{K}\times v^\perp$ and $\mathbb{K}\times v'^\perp$ as subspaces of $\mathbb{K}\times\mathbb{K}^n$ with the canonical Hermitian structure.

Proof. 1): Follows by elementary computations.
2)- (i): For $(\dot\lambda, \dot v) \in \mathbb{K} \times v^\perp$, let $(\dot\eta, \dot w) \in \mathbb{K} \times v'^\perp$ such that

$$\left(\dot\eta, \dot w\right) = \left(DF_A(\lambda, v)|_{\mathbb{K}\times v'^\perp}\right)^{-1} \cdot DF_A(\lambda, v)|_{\mathbb{K}\times v^\perp}(\dot\lambda, \dot v).$$

Then,

$$\dot\eta v + (\lambda I_n - A)\dot w = \dot\lambda v + (\lambda I_n - A)\dot v.$$

Since $(A, \lambda, v) \in \mathcal{W}$, we deduce that $\dot\eta = \dot\lambda$ and $\Pi_{v^\perp}\dot w = \dot v$. Then, we conclude that

$$\left(DF_A(\lambda, v)|_{\mathbb{K}\times v'^\perp}\right)^{-1} \cdot DF_A(\lambda, v)|_{\mathbb{K}\times v^\perp}(\dot\lambda, \dot v) = \left(\dot\lambda, (\Pi_{v^\perp}|_{v'^\perp})^{-1}(\dot v)\right).$$

Taking norms, and maximizing on the unit sphere in $\mathbb{K} \times v^\perp$, (i) follows from 1).

2)-(ii): Note that

$$
\begin{aligned}
&\| \left(DF_A(\lambda, v)|_{\mathbb{K}\times v'^{\perp}}\right)^{-1} \| \leq \\
&\qquad \| \left(DF_A(\lambda, v)|_{\mathbb{K}\times v'^{\perp}}\right)^{-1} \cdot DF_A(\lambda, v)|_{\mathbb{K}\times v^{\perp}} \| \cdot \| \left(DF_A(\lambda, v)|_{\mathbb{K}\times v^{\perp}}\right)^{-1} \|,
\end{aligned}
$$

then apply 2)-(i). □

Lemma 1.6.2. *Let $(A, \lambda, v) \in \mathbb{K}^{n\times n} \times \mathbb{K} \times \mathbb{K}^n$ such that $\|A\|_F = 1$ and $\|v\| = 1$, then $\|D^2F_A(\lambda, v)\| \leq 1$.*

Proof. Differentiating two times F_A, we get

$$
D^2F_A(\lambda, v)(\dot{\lambda}, \dot{w})(\dot{\eta}, \dot{u}) = \dot{\lambda}\dot{u} + \dot{\eta}\dot{w}, \qquad \text{for all} \quad (\dot{\lambda}, \dot{w}),\ (\dot{\eta}, \dot{u}) \in \mathbb{K} \times \mathbb{K}^n.
$$

Then,

$$
\begin{aligned}
\|D^2F_A(\lambda, v)(\dot{\lambda}, \dot{w})(\dot{\eta}, \dot{u})\| &\leq |\dot{\lambda}| \cdot \|\dot{u}\| + |\dot{\eta}| \cdot \|\dot{w}\| \\
&\leq (|\dot{\lambda}|^2 + \|\dot{u}\|^2)^{1/2} \cdot (|\dot{\eta}|^2 + \|\dot{w}\|^2)^{1/2},
\end{aligned}
$$

where the second inequality follows from Cauchy-Schwarz. □

We recall the fairly known *Neumann Lemma* (or *Banach Lemma*):

Lemma 1.6.3 (Neumann Lemma)**.** *Let $\mathbb{E}$ be a Hermitian space, and $A, I_{\mathbb{E}} : \mathbb{E} \to \mathbb{E}$ be linear operators where $I_{\mathbb{E}}$ is the identity. If $\|A - I_{\mathbb{E}}\| < 1$, then A is invertible and*

$$
\|A^{-1}\| \leq \frac{1}{1 - \|A - I_{\mathbb{E}}\|}.
$$

Proposition 1.6.1. *Let $0 < u \leq 1/(2\sqrt{2})$.*
Let $(A, \lambda, v) \in \mathcal{W}$, such that $\|A\|_F = 1$, $\|v\| = 1$, and $(\lambda_0, v_0) \in \mathbb{K} \times \mathbb{P}(\mathbb{K}^n)$. If

$$
(|\lambda_0 - \lambda|^2 + d_T(v_0, v)^2)^{1/2} < \frac{u}{\left\|DF_A\left(\lambda, v\right)|_{\mathbb{K}\times v^{\perp}}{}^{-1}\right\|},
$$

then the Newton sequence $(\lambda_k, v_k) := N_A^k(\lambda_0, v_0)$ satisfies

$$
(|\lambda_k - \lambda|^2 + d_T(v_k, v)^2)^{1/2} \leq \sqrt{2} \cdot \delta(\sqrt{2}\, u) \cdot \left(\frac{1}{2}\right)^{2^k - 1} \cdot (|\lambda_0 - \lambda|^2 + d_T(v_0, v)^2)^{1/2},
$$

for all $k > 0$, where $\delta(u) := u/(1-u)$.

Proof. Take a representative of v_0 such that $\langle v - v_0, v_0\rangle = 0$. Note that $\|v_0\| \cdot d_T(v, v_0) = \|v - v_0\|$ and $\|v_0\| \leq 1$.

In particular, the hypothesis implies that

$$\|DF_A(\lambda, v)|_{\mathbb{K}\times v^{\perp}}{}^{-1}\| \cdot \|(\lambda_0 - \lambda, v_0 - v)\| < u.$$

Taylor's expansion of F_A and DF_A in a neighborhood of (λ, v) are given by

$$F_A(\lambda', v') = DF_A(\lambda, v)(\lambda' - \lambda, v' - v) + \frac{1}{2} \cdot D^2F_A(\lambda, v)(\lambda' - \lambda, v' - v)^2, \quad (1.6.1)$$

and

$$DF_A(\lambda', v') = DF_A(\lambda, v) + D^2F_A(\lambda, v)(\lambda' - \lambda, v' - v). \quad (1.6.2)$$

One has

$$\begin{aligned}
&\Big(DF_A(\lambda, v)\big|_{\mathbb{K}\times {v_0}^{\perp}}\Big)^{-1} \cdot DF_A(\lambda_0, v_0)\big|_{\mathbb{K}\times {v_0}^{\perp}} - I_{\mathbb{K}\times {v_0}^{\perp}} = \\
&= \Big(DF_A(\lambda, v)\big|_{\mathbb{K}\times {v_0}^{\perp}}\Big)^{-1} \cdot \Big(DF_A(\lambda_0, v_0)\big|_{\mathbb{K}\times {v_0}^{\perp}} - DF_A(\lambda, v)\big|_{\mathbb{K}\times {v_0}^{\perp}}\Big) \\
&= \Big(DF_A(\lambda, v)\big|_{\mathbb{K}\times {v_0}^{\perp}}\Big)^{-1} \cdot D^2F_A(\lambda, v))(\lambda_0 - \lambda, v_0 - v)\big|_{\mathbb{K}\times {v_0}^{\perp}}.
\end{aligned}$$

Then, taking norms, we get

$$\begin{aligned}
&\left\|\Big(DF_A(\lambda, v)\big|_{\mathbb{K}\times {v_0}^{\perp}}\Big)^{-1} \cdot DF_A(\lambda_0, v_0)\big|_{\mathbb{K}\times {v_0}^{\perp}} - I_{\mathbb{K}\times {v_0}^{\perp}}\right\| \leq \\
&\leq \big\|DF_A(\lambda, v)|_{\mathbb{K}\times {v_0}^{\perp}}{}^{-1}\big\| \cdot \big\|D^2F_A(\lambda, v))(\lambda_0 - \lambda, v_0 - v)\big\| \\
&\leq \frac{1}{\cos(d_{\mathbb{P}}(v, v_0))} \cdot \big\|DF_A(\lambda, v)|_{\mathbb{K}\times v^{\perp}}{}^{-1}\big\| \cdot \|(\lambda_0 - \lambda, v_0 - v)\|,
\end{aligned}$$

where the last inequality follows from *Lemma 1.6.1* and *Lemma 1.6.2.*
In the range of angles under consideration $\|v_0\| = \cos(d_{\mathbb{P}}(v, v_0)) \geq 1/\sqrt{2}$. Then, by the condition $0 < u \leq 1/(2\sqrt{2})$ we can deduce from *Lemma 1.6.3* that

$DF_A(\lambda_0, v_0)\big|_{\mathbb{K}\times {v_0}^\perp}$ is invertible and

$$\left\| \left(DF_A(\lambda_0, v_0)\big|_{\mathbb{K}\times {v_0}^\perp} \right)^{-1} \cdot DF_A(\lambda, v)\big|_{\mathbb{K}\times {v_0}^\perp} \right\| \leq \tag{1.6.3}$$
$$\leq \frac{1}{1 - \frac{1}{\cos(d_{\mathbb{P}}(v, v_0))} \cdot \left\| DF_A(\lambda, v)|_{\mathbb{K}\times v^\perp}{}^{-1} \right\| \cdot \|(\lambda_0 - \lambda, v_0 - v)\|}.$$

Moreover,

$$\begin{aligned} N_A(\lambda_0, v_0) - (\lambda, v) \\ &= (\lambda_0 - \lambda, v_0 - v) - \left(DF_A(\lambda_0, v_0)\big|_{\mathbb{K}\times {v_0}^\perp} \right)^{-1} \cdot F_A(\lambda_0, v_0) \\ &= \left(DF_A(\lambda_0, v_0)\big|_{\mathbb{K}\times {v_0}^\perp} \right)^{-1} \cdot \\ &\qquad \cdot \left(DF_A(\lambda_0, v_0)\big|_{\mathbb{K}\times {v_0}^\perp} (\lambda_0 - \lambda, v_0 - v) - F_A(\lambda_0, v_0) \right). \end{aligned}$$

Then, from (1.6.1) we get

$$\begin{aligned} N_A(\lambda_0, v_0) - (\lambda, v) = \\ &= \frac{1}{2} \cdot \left(DF_A(\lambda_0, v_0)\big|_{\mathbb{K}\times {v_0}^\perp} \right)^{-1} \cdot D^2F_A(\lambda, v)(\lambda_0 - \lambda, v_0 - v)^2. \end{aligned}$$

Taking the canonical norm in $\mathbb{K} \times \mathbb{K}^n$, we get

$$\begin{aligned} &\|N_A(\lambda_0, v_0) - (\lambda, v)\| \leq \\ &\leq \frac{1}{2} \cdot \left\| DF_A(\lambda_0, v_0)|_{\mathbb{K}\times {v_0}^\perp}{}^{-1} \right\| \cdot \|D^2F_A(\lambda, v)(\lambda_0 - \lambda, v_0 - v)^2\|. \end{aligned}$$

Then, from (1.6.3) and *Lemma 1.6.1* we get

$$\|N_A(\lambda_0, v_0) - (\lambda, v)\| \leq$$
$$\leq \frac{\sqrt{2} \cdot \left\| DF_A(\lambda, v)|_{\mathbb{K}\times v^\perp}{}^{-1} \right\| \cdot \frac{1}{2} \cdot \|D^2F_A(\lambda, v)(\lambda_0 - \lambda, v_0 - v)^2\|.}{1 - \sqrt{2} \cdot \left\| DF_A(\lambda, v)|_{\mathbb{K}\times v^\perp}{}^{-1} \right\| \cdot \|(\lambda_0 - \lambda, v_0 - v)\|} \tag{1.6.4}$$

Therefore, from *Lemma 1.6.2*, yields

$$\|N_A(\lambda_0, v_0) - (\lambda, v)\| \leq$$
$$\leq \frac{\sqrt{2} \cdot \left\| DF_A(\lambda, v)|_{\mathbb{K}\times v^\perp}{}^{-1} \right\| \cdot \|(\lambda_0 - \lambda, v_0 - v)\|}{1 - \sqrt{2} \cdot \left\| DF_A(\lambda, v)|_{\mathbb{K}\times v^\perp}{}^{-1} \right\| \cdot \|(\lambda_0 - \lambda, v_0 - v)\|} \cdot \frac{1}{2} \|(\lambda_0 - \lambda, v_0 - v)\|.$$

Then,

$$\|N_A(\lambda_0, v_0) - (\lambda, v)\| \leq$$
$$\leq \frac{\sqrt{2} \cdot \left\| DF_A(\lambda, v)|_{\mathbb{K}\times v^\perp}{}^{-1} \right\| \cdot (|\lambda_0 - \lambda|^2 + d_T(v_0, v)^2)^{1/2}}{1 - \sqrt{2} \cdot \left\| DF_A(\lambda, v)|_{\mathbb{K}\times v^\perp}{}^{-1} \right\| \cdot (|\lambda_0 - \lambda|^2 + d_T(v_0, v)^2)^{1/2}} \cdot$$
$$\cdot \frac{1}{2} (|\lambda_0 - \lambda|^2 + d_T(v_0, v)^2)^{1/2}.$$

Let $(\lambda_1, v_1) := N_A(\lambda_0, v_0)$.
From the proof of *Lemma 1.6.2* we have $D^2 F_A(\lambda, v)(\lambda_0 - \lambda, v_0 - v)^2 = 2(\lambda_0 - \lambda)(v_0 - v)$, then, from (1.6.4) one can deduce that $\|v_1 - v\| < \delta(\sqrt{2}\, u)\|v_0 - v\|$, where $\delta(u) = u/(1-u)$. Since $u \leq 1/(2\sqrt{2})$, we have $\delta(\sqrt{2}\, u) \leq 1$, then from *Lemma 2, (4)* of Blum *et al.* [1998] (page 264) we get

$$d_T(v_1, v) \leq \frac{\|v_1 - v\|}{\|v_0\|} \leq \sqrt{2} \cdot \|v_1 - v\|.$$

Hence

$$(|\lambda_1 - \lambda|^2 + d_T(v_1, v)^2)^{1/2} \leq$$
$$\leq \frac{2 \cdot \left\| DF_A(\lambda, v)|_{\mathbb{K}\times v^\perp}{}^{-1} \right\| \cdot (|\lambda_0 - \lambda|^2 + d_T(v_0, v)^2)^{1/2}}{1 - \sqrt{2} \cdot \left\| DF_A(\lambda, v)|_{\mathbb{K}\times v^\perp}{}^{-1} \right\| \cdot (|\lambda_0 - \lambda|^2 + d_T(v_0, v)^2)^{1/2}} \cdot$$
$$\cdot \frac{1}{2} (|\lambda_0 - \lambda|^2 + d_T(v_0, v)^2)^{1/2}. \quad (1.6.5)$$

Therefore,

$$(|\lambda_1 - \lambda|^2 + d_T(v_1, v)^2)^{1/2} \leq \sqrt{2} \cdot \delta(\sqrt{2}\, u) \cdot \frac{1}{2} (|\lambda_0 - \lambda|^2 + d_T(v_0, v)^2)^{1/2}. \quad (1.6.6)$$

From (1.6.6), (1.6.5), and the fact that $\delta(\sqrt{2}\, u) \leq 1$, working by induction we get

$$(|\lambda_k - \lambda|^2 + d_T(v_k, v)^2)^{1/2} \le \sqrt{2} \cdot \delta(\sqrt{2}\, u) \cdot \left(\frac{1}{2}\right)^{2^k - 1} \cdot (|\lambda_0 - \lambda|^2 + d_T(v_0, v)^2)^{1/2},$$

for all $k > 0$, where $(\lambda_k, v_k) := N_A^k(\lambda_0, v_0)$. □

Proposition 1.6.2. *Let* $(A, \lambda, v) \in \mathcal{W}$, *such that* $\|A\|_F = 1$ *and* $\|v\| = 1$. *Then,*

$$\mu(A, \lambda, v) \le \|DF_A\,(\lambda, v)\,|_{\mathbb{K}\times v^\perp}{}^{-1}\| \le 2 \cdot \mu(A, \lambda, v).$$

Proof. Since the action of $\mathbb{U}_n(\mathbb{K})$ on $\mathbb{P}(\mathbb{K}^n)$ is transitive, we may assume that $v = (1, 0, \ldots, 0)^T$. Then, completing to a basis of $\mathbb{K} \times v^\perp$, we have that

$$A = \begin{pmatrix} \lambda & w \\ 0 & \hat{A} \end{pmatrix}, \quad DF_A(\lambda, v)|_{\mathbb{K}\times v^\perp} = \begin{pmatrix} 1 & -w \\ 0 & \Pi_{v^\perp}(\lambda I_n - A)|_{v^\perp} \end{pmatrix},$$

where $w \in \mathbb{K}^{1\times(n-1)}$.

Note that $(DF_A(\lambda, v)|_{\mathbb{K}\times v^\perp})^{-1} = \begin{pmatrix} 1 & w(\Pi_{v^\perp}(\lambda I_n - A)|_{v^\perp})^{-1} \\ 0 & (\Pi_{v^\perp}(\lambda I_n - A)|_{v^\perp})^{-1} \end{pmatrix}$. Hence

$$\|DF_A\,(\lambda, v)\,|_{\mathbb{K}\times v^\perp}{}^{-1}\| \ge \max\{1, \|(\Pi_{v^\perp}(\lambda I_n - A)|_{v^\perp})^{-1}\|\} = \mu(A, \lambda, v).$$

On the other hand,

$$\begin{aligned}
&\|DF_A\,(\lambda, v)\,|_{\mathbb{K}\times v^\perp}{}^{-1}\| \le \\
\le\ & \left\|\begin{pmatrix} 1 & w(\Pi_{v^\perp}(\lambda I_n - A)|_{v^\perp})^{-1} \\ 0 & 0 \end{pmatrix}\right\| + \left\|\begin{pmatrix} 0 & 0 \\ 0 & (\Pi_{v^\perp}(\lambda I_n - A)|_{v^\perp})^{-1} \end{pmatrix}\right\| \\
\le\ & \max\{1, \|w(\Pi_{v^\perp}(\lambda I_n - A)|_{v^\perp})^{-1}\|\} + \|(\Pi_{v^\perp}(\lambda I_n - A)|_{v^\perp})^{-1}\| \\
\le\ & 2 \cdot \mu(A, \lambda, v).
\end{aligned}$$

□

Proof of *Proposition 1.4.1*

Proof of Proposition 1.4.1. The proof follows directly from *Proposition 1.6.1*, *Proposition 1.6.2* and *Lemma 1.3.5.* □

Proof of Lemma 1.4.2

Lemma 1.6.4. *Let $A \in \mathbb{K}^{n\times n}$, $A \neq 0_n$, such that $\|A\|_F = 1$. Let λ, $\lambda' \in \mathbb{K}$ such that $|\lambda| \leq 1$.*

1. *If $|\lambda' - \lambda| \leq c$ for some $0 \leq c < \sqrt{2}$, then, there exists $\beta_c > 1$ such that*

$$d_{\mathbb{P}}((A,\lambda),(A,\lambda)) \leq \beta_c \cdot |\lambda' - \lambda|.$$

   One may choose $\beta_c = (1 - c^2/2)^{-1/2}$.

2. *If $d_{\mathbb{P}}((A,\lambda),(A,\lambda')) \leq \hat{\theta}$ for some $0 \leq \hat{\theta} < \pi/4$, then, there exist $R_\theta > 1$ such that*

$$|\lambda' - \lambda| \leq R_{\hat{\theta}} \cdot d_{\mathbb{P}}((A,\lambda),(A,\lambda')).$$

   One may choose $R_{\hat{\theta}} = [\sqrt{2}/\cos(\hat{\theta} + \pi/4)^3]^{1/2}$.

Proof. Let $\theta := d_{\mathbb{P}}((A,\lambda),(A,\lambda'))$. By the law of cosines we know that

$$|\lambda - \lambda'|^2 = 1 + |\lambda|^2 + 1 + |\lambda'|^2 - 2 \cdot \sqrt{1+|\lambda|^2} \cdot \sqrt{1+|\lambda'|^2} \cdot \cos\theta.$$

Then,

$$\begin{aligned} |\lambda - \lambda'|^2 \quad = \quad & \left(\sqrt{1+|\lambda|^2} - \sqrt{1+|\lambda'|^2}\right)^2 + \\ & +2 \cdot \sqrt{1+|\lambda|^2} \cdot \sqrt{1+|\lambda|^2} \cdot (1 - \cos\theta). \end{aligned} \tag{1.6.7}$$

From (1.6.7) we get that

$$|\lambda - \lambda'|^2 \geq 2 \cdot \sqrt{1+|\lambda|^2} \cdot \sqrt{1+|\lambda|^2} \cdot (1 - \cos\theta),$$

i.e.

$$1 - \cos\theta \leq \frac{|\lambda - \lambda'|^2}{2 \cdot \sqrt{1+|\lambda|^2} \cdot \sqrt{1+|\lambda|^2}} \leq \frac{|\lambda' - \lambda|^2}{2}. \tag{1.6.8}$$

Therefore, $1 - \cos\theta \leq \frac{c^2}{2}$, and hence the angle θ is bounded above by $\arccos(1 - c^2/2)$. By the Taylor expansion of cosine near 0 we get the bound

$$\theta^2 \leq \frac{2}{1 - c^2/2} \cdot (1 - \cos\theta).$$

Then, from (1.6.8) we can deduce the upper bound in (1).

For the lower bound in (2), we rewrite the cosine law and we get:

$$\begin{aligned}|\lambda-\lambda'|^2 &= \left(\frac{|\lambda|^2-|\lambda'|^2}{\sqrt{1+|\lambda|^2}+\sqrt{1+|\lambda'|^2}}\right)^2+\\ &\qquad +2\sqrt{1+|\lambda|^2}\cdot\sqrt{1+|\lambda'|^2}\cdot(1-\cos\theta).\end{aligned}$$

Since $||\lambda|-|\lambda'||\leq|\lambda-\lambda'|$ and $1-\cos\theta\leq\theta^2/2$, then,

$$\begin{aligned}|\lambda-\lambda'|^2\leq\left(\frac{|\lambda|+|\lambda'|}{\sqrt{1+|\lambda|^2}+\sqrt{1+|\lambda'|^2}}\right)^2\cdot|\lambda-\lambda'|^2+\\ +\sqrt{1+|\lambda|^2}\cdot\sqrt{1+|\lambda'|^2}\cdot\theta^2\end{aligned} \tag{1.6.9}$$

Since $0\leq|\lambda|\leq 1$, is easy to see that

$$\frac{|\lambda|+|\lambda'|}{\sqrt{1+|\lambda|^2}+\sqrt{1+|\lambda'|^2}}\leq\frac{1+|\lambda'|}{\sqrt{2}+\sqrt{1+|\lambda'|^2}}.$$

Moreover, by elementary arguments one can see that $|\lambda'|\leq\tan(\hat{\theta}+\pi/4)$, and therefore one can get

$$\begin{aligned}\frac{|\lambda|+|\lambda'|}{\sqrt{1+|\lambda|^2}+\sqrt{1+|\lambda'|^2}} &\leq \frac{1+\tan(\hat{\theta}+\pi/4)}{\sqrt{2}+\sqrt{1+\tan(\hat{\theta}+\pi/4)^2}}\\ &\leq \frac{\tan(\hat{\theta}+\pi/4)}{\sqrt{1+\tan(\hat{\theta}+\pi/4)^2}}=\sin(\hat{\theta}+\pi/4).\end{aligned}$$

Then, from (1.6.9),

$$|\lambda-\lambda'|^2\leq\frac{\sqrt{1+|\lambda|^2}\cdot\sqrt{1+|\lambda'|^2}}{\cos(\hat{\theta}+\pi/4)^2}\cdot\theta^2,$$

and hence

$$|\lambda-\lambda'|^2\leq\frac{\sqrt{2}}{\cos(\hat{\theta}+\pi/4)^3}\cdot\theta^2.$$

□

Remark 1.6.2. Note that if $(A, \lambda) \in \pi_1(\mathcal{V}) \subset \mathbb{P}(\mathbb{K}^{n\times n} \times \mathbb{K})$ then $|\lambda| \leq \|A\|_F$ is always satisfied.

Proof of Lemma 1.4.2

Proof of Lemma 1.4.2. The proof of (1) and (2) follows directly from de definition of $d_{\mathbb{P}^2}$ and *Lemma 1.6.4.* □

Chapter 2

Complexity of The Eigenvalue Problem II: Distance Estimates in the Condition Metric

2.1 Introduction

Following *Chapter 1*, we define the solution variety as

$$\mathcal{V} =: \left\{(A, \lambda, v) \in \mathbb{P}\left(\mathbb{K}^{n\times n} \times \mathbb{K}\right) \times \mathbb{P}\left(\mathbb{K}^n\right) : \; (\lambda I_n - A)v = 0\right\},$$

where $\mathbb{P}(\mathbb{E})$ denotes the projective space associated with the vector space $\mathbb{E}$.

Recall that $\mathcal{W} \subset \mathcal{V}$ be the set of *well-posed* problems, that is the set of triples $(A, \lambda, v) \in \mathcal{V}$ such that λ is a simple eigenvalue. In that case, for a fixed representative $(A, \lambda, v) \in \mathcal{V}$, the operator $\Pi_{v^\perp}(\lambda I_n - A)|_{v^\perp}$ is invertible, where $\Pi_{v^\perp}$ denotes the orthogonal projection of $\mathbb{K}^n$ onto $v^\perp$. The condition number of (A, λ, v) is defined by

$$\mu(A, \lambda, v) := \max\left\{1, \|A\|_F \cdot \|\Pi_{v^\perp}(\lambda I_n - A)|_{v^\perp}{}^{-1}\|\right\}, \tag{2.1.1}$$

where $\|\cdot\|_F$ and $\|\cdot\|$ are the Frobenius and operator norms in the space of matrices. We also let $\mu(A, \lambda, v) = \infty$ when $(A, \lambda, v) \in \mathcal{V} - \mathcal{W}$.

When $\Gamma(t)$, $a \le t \le b$, is an absolutely continuous path in $\mathcal{W}$, we defined in last chapter its *condition-length* as

$$\ell_\mu(\Gamma) := \int_a^b \left\|\dot{\Gamma}(t)\right\|_{\Gamma(t)} \cdot \mu\left(\Gamma(t)\right)\, dt, \tag{2.1.2}$$

where $\left\|\dot{\Gamma}(t)\right\|_{\Gamma(t)}$ is the norm of $\dot{\Gamma}(t)$ in the unitarily invariant Riemannian structure on $\mathcal{V}$ (see *Section* 2.1.1). Here , $\dot{\Gamma}(t) := \Pi_{\Gamma(t)^\perp} \frac{d}{dt}\Gamma(t)$, where $\frac{d}{dt}\Gamma(t)$ is the "free" derivative.

Recall *Theorem 2* from last chapter:

> *There is a universal constant $C > 0$ such that for any absolutely continuous path Γ in $\mathcal{W}$, there exists a sequence which approximates Γ, and such that the complexity of the sequence is*
>
> $$K \leq C\, \ell_\mu(\Gamma) + 1.$$
>
> *(One may choose $C = 120$).*

2.1.1 Main Theorem

Let $\{e_1, \ldots, e_n\}$ be the canonical basis of $\mathbb{K}^n$, and $G := e_1 \cdot e_1^* \in \mathbb{K}^{n\times n}$. Let $\mathcal{W}_0$ be the set of problems $(A, \lambda, v) \in \mathcal{W}$ such that $\mu(A, \lambda, v) = 1$. Notice that $(G, 1, e_1) \in \mathcal{W}_0$.

Theorem 4. *For every problem $(A, \lambda, v) \in \mathcal{W}$ there exist a path Γ in $\mathcal{W}$ joining (A, λ, v) with $(G, 1, e_1)$, and such that*

$$\ell_\mu(\Gamma) \quad \leq \sqrt{2}\sqrt{2n+1} \cdot \left(1 + \log\left(\sqrt{2}\,\mu(A, \lambda, v)\right)\right) + \pi\sqrt{n-1} + \sqrt{n+1} + \pi\sqrt{2n}.$$

Canonical Metric Structures

In this section we recall the canonical metric structures.

The space $\mathbb{K}^n$ is equipped with the canonical Hermitian inner product $\langle \cdot, \cdot \rangle$. The space $\mathbb{K}^{n\times n}$ is equipped with the Frobenius Hermitian inner product

$$\langle A, B \rangle_F := \text{trace}\ (B^* A),$$

where B^* denotes the adjoint of B.

In general, if $\mathbb{E}$ is a finite dimensional vector space over $\mathbb{K}$ with the Hermitian inner product $\langle \cdot, \cdot \rangle$, we can define an Hermitian structure over $\mathbb{P}(\mathbb{E})$ in the following way: for $x \in \mathbb{E}$,

$$\langle w, w' \rangle_x := \frac{\langle w, w' \rangle}{\|x\|^2},$$

for all w, w' in the Hermitian complement $x^\perp$ of x in $\mathbb{E}$, which is a natural representative of the tangent space $T_x\mathbb{P}(\mathbb{E})$.

In this way, the space $\mathbb{P}\big(\mathbb{K}^{n\times n} \times \mathbb{K}\big) \times \mathbb{P}(\mathbb{K}^n)$ inherits the Hermitian product structure

$$\|(\dot{A}, \dot{\lambda}, \dot{v})\|^2_{(A,\lambda,v)} = \|(\dot{A}, \dot{\lambda})\|^2_{(A,\lambda)} + \|\dot{v}\|^2_v \tag{2.1.3}$$

for all $(\dot{A}, \dot{\lambda}, \dot{v}) \in (A, \lambda)^\perp \times v^\perp$.

Let $\mathbb{U}_n(\mathbb{K})$ stand for the unitary group when $\mathbb{K} = \mathbb{C}$ or the orthogonal group when $\mathbb{K} = \mathbb{R}$. The group $\mathbb{U}_n(\mathbb{K})$ acts on $\mathbb{P}(\mathbb{K}^n)$ in the natural way, and acts on $\mathbb{K}^{n\times n}$ by sending $A \mapsto UAU^{-1}$. Moreover if $(A, \lambda, v) \in \mathcal{V}$, then $(UAU^{-1}, \lambda, Uv) \in \mathcal{V}$. Thus, $\mathcal{V}$ is invariant under the product action $\mathbb{U}_n(\mathbb{K}) \times \mathcal{V} \to \mathcal{V}$ given by

$$U \cdot (A, \lambda, v) \mapsto (UAU^{-1}, \lambda, Uv), \quad U \in \mathbb{U}_n(\mathbb{K}).$$

The group $\mathbb{U}_n(\mathbb{K})$ preserves the Hermitian structure on $\mathcal{V}$, therefore $\mathbb{U}_n(\mathbb{K})$ acts by isometries on $\mathcal{V}$. Moreover, the condition number μ is $\mathbb{U}_n(\mathbb{K})$-invariant.

2.2 Proof of Main Theorem

Proposition 2.2.1. *Let $(A, \lambda, v) \in \mathcal{W}$. Then, there exists $\Gamma(t) = (A(t), \lambda(t), v(t)) \in \mathcal{W}$, such that*

- $\Gamma(0) = (A, \lambda, v)$*;* $\Gamma(1) = (B, 0, v)$.
- *B has v as a left and right eigenvector;*
- $\|B\|_F^{-1} \cdot \Pi_{v^\perp} B|_{v^\perp} : v^\perp \to v^\perp$ *is a linear isometry, and*
- $$\ell_\mu(\Gamma) \le \sqrt{2}\sqrt{2n+1}\left(1 + \log\left(\sqrt{2}\,\mu(A, \lambda, v)\right)\right).$$

For the proof of *Proposition 2.2.1* we use the following lemma.

Lemma 2.2.1. *Let* $0 < \sigma < 1$. *Then,*

$$\int_0^1 \frac{1}{(1-t)\sigma + t}\, dt = \frac{\log(\frac{1}{\sigma})}{1-\sigma} \leq 1 + \log(\frac{1}{\sigma})$$

Proof. The equality is straightforward.
Since the Taylor expansion of $\log(1-x) = -\sum_{n=1}^{+\infty} \frac{x^n}{n}$, we have

$$\frac{-\log(\sigma)}{1-\sigma} = \frac{-\log(1-(1-\sigma))}{1-\sigma} = \sum_{n=1}^{+\infty} \frac{(1-\sigma)^{n-1}}{n} \leq \sum_{n=1}^{+\infty} \frac{(1-\sigma)^{n-1}}{n-1} = 1 - \log(\sigma).$$

□

Lemma 2.2.2. *Let* $(A, \lambda, v) \in \mathcal{W}$. *Then,* $\|A\|_F \cdot \|\Pi_{v^\perp}(\lambda I_n - A)|_{v^\perp}{}^{-1}\| \geq 1/\sqrt{2}$. *In particular* $\mu(A, \lambda, v) \leq \sqrt{2}\|A\|_F \cdot \|\Pi_{v^\perp}(\lambda I_n - A)|_{v^\perp}{}^{-1}\|$.

Proof. One has,

$$\|\Pi_{v^\perp}(\lambda I_n - A)|_{v^\perp}\| \leq \|\Pi_{v^\perp}(A)|_{v^\perp}\| + |\lambda| \leq \sqrt{2}\|A\|_F,$$

that is, $\|\Pi_{v^\perp}(\lambda I_n - A)|_{v^\perp}\| \leq \sqrt{2}\|A\|_F$. Therefore,

$$\begin{aligned} 1 &= \|\, (\Pi_{v^\perp}(\lambda I_n - A)|_{v^\perp})^{-1}\, \Pi_{v^\perp}(\lambda I_n - A)|_{v^\perp}\| \\ &\leq \sqrt{2}\|A\|_F\|\, (\Pi_{v^\perp}(\lambda I_n - A)|_{v^\perp})^{-1}\, \|. \end{aligned}$$

Therefore, we conclude that for $(A, \lambda, v) \in \mathcal{W}$

$$\|A\|_F \cdot \|\Pi_{v^\perp}(\lambda I_n - A)|_{v^\perp}{}^{-1}\| \leq \mu(A, \lambda, v) \leq \sqrt{2}\|A\|_F \cdot \|\Pi_{v^\perp}(\lambda I_n - A)|_{v^\perp}{}^{-1}\|.$$

□

Proof of Proposition 2.2.1. Fix a representative of $(A, \lambda, v) \in \mathcal{W}$. Without loss of generality we may assume $v = e_1$. Moreover, since our framework is scale invariant in (A, λ), we may assume also that $\|A\|_F = 1$. In this case, we have

$$A = \begin{pmatrix} \lambda & A_1 \\ 0 & \hat{A} \end{pmatrix},$$

where, in particular, $|\lambda| \leq \|A\|_F = 1$.

Since $(A, \lambda, v) \in \mathcal{W}$, there exists $U, V \in \mathbb{U}_{n-1}(\mathbb{K})$ such that $\hat{A} - \lambda I_{n-1} = UDV^*$, where $D = \operatorname{diag}(\sigma_2, \ldots, \sigma_n)$, $0 < \sigma_n \leq \sigma_{n-1} \leq \ldots \sigma_2$.

Then, from *Lemma 2.2.2* we get

$$\frac{1}{\sigma_n} \leq \mu(A, \lambda, v) \leq \sqrt{2} \cdot \frac{1}{\sigma_n}.$$

For $t \in [0, 1]$, let

$$A(t) = (1-t)A + t\begin{pmatrix} 0 & 0 \\ 0 & UV^* \end{pmatrix} = \begin{pmatrix} (1-t)\lambda & (1-t)A_1 \\ 0 & (1-t)(\lambda I_{n-1} + UDV^*) + tUV^* \end{pmatrix},$$

and let $\Gamma(t) = (A(t), (1-t)\lambda, e_1) \in \mathcal{V}$. Note that $\Gamma(1) = \left(\begin{pmatrix} 0 & 0 \\ 0 & UV^* \end{pmatrix}, 0, e_1\right)$ satisfy the first three conditions.

Since

$$\mu(\Gamma(t)) \leq \sqrt{2}\|A(t)\|_F \cdot \left\|((1-t)D + tI_{n-1})^{-1}\right\| = \sqrt{2}\frac{\|A(t)\|_F}{(1-t)\sigma_n + t} < +\infty,$$

then $\Gamma(t) \in \mathcal{W}$,

Taking the free derivative with respect to t we get

$$\frac{d}{dt}\Gamma(t) = \left(\begin{pmatrix} 0 & 0 \\ 0 & UV^* \end{pmatrix} - A, -\lambda, 0\right).$$

Therefore

$$\left\|\dot{\Gamma}(t)\right\|_{\Gamma(t)} \leq \frac{((\|UV^*\|_F + \|A\|_F)^2 + |\lambda|^2)^{1/2}}{(\|A(t)\|_F^2 + |(1-t)\lambda|^2)^{1/2}} \leq \frac{\sqrt{2n+1}}{(\|A(t)\|_F^2 + |(1-t)\lambda|^2)^{1/2}}.$$

Hence,

$$\begin{aligned}
\ell_\mu(\Gamma) &= \int_0^1 \left\|\dot\Gamma(t)\right\|_{\Gamma(t)} \cdot \mu(\Gamma(t))\, dt \\
&\leq \sqrt{2}\cdot \int_0^1 \frac{\sqrt{2n+1}}{(\|A(t)\|_F^2 + |(1-t)\lambda|^2)^{1/2}} \cdot \frac{\|A(t)\|_F}{(1-t)\sigma_n + t}\, dt \\
&\leq \sqrt{2}\cdot\sqrt{2n+1}\int_0^1 \frac{1}{(1-t)\frac{\sigma_n}{\sqrt{2}} + t}\, dt.
\end{aligned}$$

Since $\frac{\sigma_n}{\sqrt{2}} \in (0,1)$, we get from *Lemma 2.2.1* that

$$\ell_\mu(\Gamma) \leq \sqrt{2}\sqrt{2n+1}\left(1 + \log\left(\sqrt{2}\,\mu(A,\lambda,v)\right)\right).$$

□

Lemma 2.2.3. *Let $(B,0,v) \in \mathcal{W}$ such that B has v as a left and right eigenvector, and $\|B\|_F{}^{-1}\Pi_{v^\perp}B|_{v^\perp}$ is a linear isometry of $v^\perp$ onto itself. Then, there exist a path $\Gamma_2 : [0,1] \to \mathcal{W}$, starting at $(B,0,v)$ such that*

- *$\Gamma_2(1) = (C,0,v)$;*
- *C has v as a left and right eigenvector;*
- *$\|C\|_F{}^{-1} \cdot \Pi_{v^\perp}C|_{v^\perp} = I_{v^\perp}$ is the identity operator, and*
- $$\ell_\mu(\Gamma_2) \leq \pi \cdot \sqrt{n-1}.$$

Proof. Without loss of generality, we may assume $v = e_1$, and $\|B\|_F = 1$.
Then, $B = \begin{pmatrix} 0 & 0 \\ 0 & U \end{pmatrix}$, where $U \in \mathbb{U}_{n-1}(\mathbb{K})$. There exists $V \in \mathbb{U}_{n-1}$ such that $U = V\mathrm{diag}(e^{i\theta_2},\ldots,e^{i\theta_n})V^{-1}$, for $\theta_2,\ldots,\theta_n \in [-\pi,\pi]$. Let

$$U(t) = V\mathrm{diag}(e^{(1-t)i\theta_2},\ldots,e^{(1-t)i\theta_n})V^{-1}, \quad 0 \leq t \leq 1.$$

Define $\Gamma_2(t) = (B(t),0,v) \in \mathcal{W}$ where $B(t) = \begin{pmatrix} 0 & 0 \\ 0 & U(t) \end{pmatrix}$. Note that $\Gamma(1)$ satisfy the first three conditions of the lemma.

Then,

$$\mu(\Gamma_2(t)) = \|U(t)\|_F = \sqrt{n-1}.$$

Note that $\frac{d}{dt}\Gamma_2(t) = \left(\begin{pmatrix} 0 & 0 \\ 0 & \dot{U}(t) \end{pmatrix}, 0, 0\right)$, where $\dot{U}(t)$ is an antisymmetric matrix. Then $\langle \frac{d}{dt}\Gamma_2(t), \Gamma_2(t)\rangle = 0$, and therefore $\frac{d}{dt}\Gamma_2(t) = \dot{\Gamma}_2(t)$. Then,

$$\|\dot{\Gamma}_2(t)\|_{\Gamma_2(t)} = \frac{\|\dot{U}(t)\|_F}{\|B(t)\|_F} = \frac{(|\theta_2|^2 + \ldots + |\theta_n|^2)^{1/2}}{\sqrt{n-1}} \leq \pi.$$

Then,

$$\ell_\mu(\Gamma_2) = \int_0^1 \|\dot{\Gamma}_2(t)\|_{\Gamma_2(t)} \cdot \mu(\Gamma_2(t))\, dt = \pi\sqrt{n-1}.$$

□

Lemma 2.2.4. *Let $(C, 0, v) \in \mathcal{W}$, such that C has v as a left and right eigenvector, and $\|C\|_F^{-1} \cdot \Pi_{v^\perp} C|_{v^\perp} : v^\perp \to v^\perp$ is the identity operator. Then, there exist a path $\Gamma_3 : [0,1] \to \mathcal{W}$, joining $(C, 0, v)$with $(\frac{vv^*}{\|v\|^2}, 1, \frac{v}{\|v\|})$, and*

$$\ell_\mu(\Gamma_3) \leq \sqrt{n+1}.$$

Proof. Assume that $v = e_1$ and $\|C\|_F = 1$. Moreover, since our framework is scale invariant, multiplying by -1, we may assume also that $C = \begin{pmatrix} 0 & 0 \\ 0 & -I_{n-1} \end{pmatrix}$. For $t \in [0,1]$, let $\Gamma_3(t) = ((1-t)C + te_1^* e_1, t, e_1)$. Note that $\Gamma_3(1) = (e_1^* e_1, 1, e_1)$. One has $\frac{d}{dt}\Gamma_3(t) = (I_n, 1, 0)$ and $\mu(\Gamma_3(t)) = \|(1-t)C + te_1^* e_1\|_F$. Then, $\|\dot{\Gamma}(t)\| \leq \sqrt{n+1}$ and we conclude

$$\ell_\mu(\Gamma_3) \leq \sqrt{n+1}.$$

□

Lemma 2.2.5. *Let $(\frac{vv^*}{\|v\|^2}, 1, \frac{v}{\|v\|}) \in \mathcal{W}_0$. Then there exist a path $\Gamma_4 : [0,1] \to \mathcal{W}_0$ joining $(\frac{vv^*}{\|v\|^2}, 1, \frac{v}{\|v\|})$ with $(G, 1, e_1)$ such that*

$$\ell_\mu(\Gamma_4) \leq \sqrt{\pi^2 n + 1}.$$

Proof. Let v be a representantive of norm 1, and $U \in \mathbb{U}_n(\mathbb{K})$ such that $Uv = e_1$. There exists $V \in \mathbb{U}_n(\mathbb{K})$ and real numbers $\theta_1, \ldots, \theta_n \in [-\pi, \pi]$ such that

$U = V\mathrm{diag}(e^{i\theta_1}, \ldots, e^{i\theta_n})V^{-1}$. Let $U(t) = V\mathrm{diag}(e^{it\theta_1}, \ldots, e^{it\theta_n})V^{-1}$ and $\Gamma_4(t) = U(t) \cdot (vv^*, 1, v)$ for $t \in [0,1]$. By the invariance of μ under the action of $\mathbb{U}_n(\mathbb{K})$, we have $\Gamma_4(t) \in \mathcal{W}_0$ for $t \in [0,1]$. Moreover, $\langle \frac{d}{dt}\Gamma_4(t), \Gamma_4(t)\rangle = 0$, therefore $\frac{d}{dt}\Gamma_4(t) = \dot{\Gamma}_4(t)$, and

$$\begin{aligned}\left\|\dot{\Gamma}_4(t)\right\|^2_{\Gamma_4(t)} &= \frac{\|\dot{U}(t)vv^*U(t)^* + U(t)vv^*\dot{U}(t)^*\|_F^2}{\|U(t)vv^*U_T^*\|_F^2 + 1} + \frac{\|\dot{U}(t)v\|^2}{\|v\|^2} \\ &= 2\|\dot{U}(t)v\|^2,\end{aligned}$$

where we use the fact that $\langle \dot{U}(t)vv^*U(t)^*, U(t)vv^*\dot{U}(t)^*\rangle_F = 0$. Since $\|\dot{U}(t)v\| \leq \|\dot{U}(t)\|_F \leq \pi\sqrt{n}$, we obtain

$$\left\|\dot{\Gamma}_4(t)\right\|_{\Gamma_4(t)} \leq \pi\sqrt{2n}.$$

□

Proof of Theorem 4. The proof follows from *Proposition 2.2.1*, *Lemma 2.2.3*, *Lemma 2.2.4* and *Lemma 2.2.5*. □

Chapter 3

Smale's Fundamental Theorem of Algebra reconsidered

In his 1981 Fundamental Theorem of Algebra paper Steve Smale initiated the complexity theory of finding a solution of polynomial equations of one complex variable by a variant of Newtons's method. In this chapter we reconsider his algorithm in the light of work done in the intervening years. The main theorem raises more problems than it solves. This chapter follows from a joint work with Michael Shub (c.f. Armentano & Shub [2012]).

3.1 Introduction and Main Result

In his paper [Smale, 1981] Steve Smale initiated the complexity theory of finding a solution of polynomial equations of one complex variable by a variant of Newtons's method. More specifically he considered the space $\mathscr{P}_d$ of monic polynomials of degree d,

$$f(z) = \sum_{i=0}^{d} a_i z^i, \quad a_d = 1 \quad \text{and} \quad a_i \in \mathbb{C}, \qquad (i = 0, \dots, d-1).$$

He identified $\mathscr{P}_d$ with $\mathbb{C}^d$, with coordinates $(a_0, \ldots, a_{d-1}) \in \mathbb{C}^d$. In $\mathscr{P}_d$ he considered the poly-cylinder

$$\mathcal{P}_1 = \{f \in \mathscr{P}_d : |a_i| < 1, \ i = 0, \ldots, d-1\}$$

to have finite volume and he obtained a probability space by normalizing the volume equal 1. The algorithm he analyzed is given by: let $0 < h \leq 1$ and let $z_0 = 0$. Inductively define $z_n = T_h(z_{n-1})$ where T_h is the modified Newton's method for f given by $T_h(z) = z - h\frac{f(z)}{f'(z)}$.

His eponymous main theorem was:

> Main Theorem: *There is a universal polynomial $S(d, 1/\mu)$ and a function $h = h(d, \mu)$ such that for degree d and μ, $0 < \mu < 1$, the following is true with probability $1 - \mu$. Let $x_0 = 0$. Then $x_n = T_h(x_{n-1})$ is defined for all $n > 0$ and x_s is an approximate zero for f where $s = S(d, 1/\mu)$.*

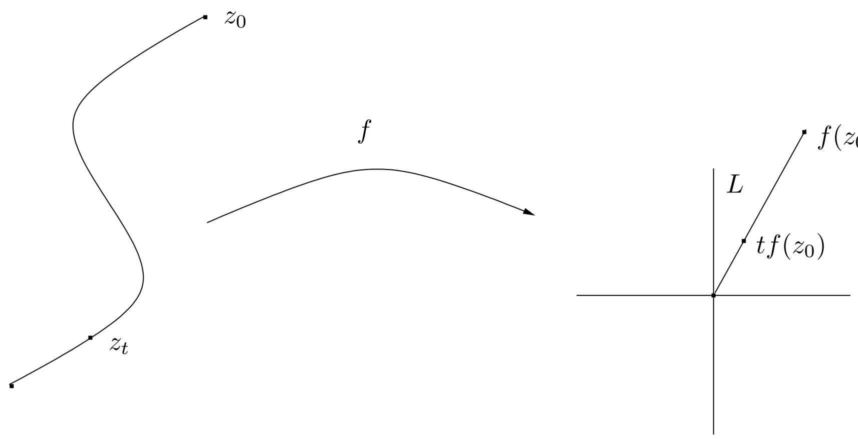
In Smale [1981], that x_s is an approximate zero meant that there is an x^* such that $f(x^*) = 0$, $x_n \to x^*$ and $\frac{|f(x_{j+1})|}{|f(x_j)|} < \frac{1}{2}$, for $j \geq s$, where $x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$. That is, x_{k+1} is defined by the usual Newton's method for f. Smale mentions that the polynomial S may be taken to be $\frac{100(d+2)^d}{\mu^7}$. The notion of approximate zero was changed in later papers (see Blum *et al.* [1998] for the later version). The new version incorporates immediate quadratic convergence of Newton's method on an approximate zero. In the remainder of this chapter an approximate zero refers to the new version.

Note that $\frac{1}{\mu^7}$ is not finitely integrable, so Smale's initial algorithm was not proven to be finite average time or cost when the upper bound is averaged over $\mathcal{P}_1$ (see [Blum *et al.*, 1998, page 208, Proposition 2]).

A tremendous amount of work has been done in the last 30 years following on Smale's initial contribution, much too much to survey here. Let us mention a few of the main changes. In one variable a lot of work has been done concerning the choice of good starting point z_0 for Smale's algorithm other than zero. See chapters 8 and 9 of Blum *et al.* [1998] and references in the commentary on chapter 9. The latest work in this direction is Kim *et al.* [2011].

Smale's algorithm may be given the following interpretation. For $z_0 \in \mathbb{C}$, consider $f_t = f - (1-t)f(z_0)$, for $0 \le t \le 1$. f_t is a polynomial of the same degree as f, z_0 is a zero of f_0 and $f_1 = f$. So, we analytically continue z_0 to z_t a zero of f_t. For $t = 1$ we arrive at a zero of f. Newton's method is then used to produce a discrete numerical approximation to the path (f_t, z_t).

If we view f as a mapping from $\mathbb{C}$ to $\mathbb{C}$, then the curve z_t is the branch of the inverse image of the line segment $L = \{tf(z_0) :\ 0 \le t \le 1\}$, containing z_0.



Here are several of the changes made in the intervening years. Renegar [1987] considered systems of n-complex polynomial in n-variables. Given a degree d, we let $\mathscr{P}_d$ stands for the vector space of degree d polynomials in n complex variables

$$\mathscr{P}_d = \{f :\ f(x) = \sum_{\|\alpha\|=d} a_\alpha x^\alpha\}$$

where $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ is a multi-index, $\|\alpha\| = \sum_{k=1}^{d} \alpha_k$, $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, $a_\alpha \in \mathbb{C}$. We have suppressed de n for case of notation. It should be understood from the context.

For $(d) = (d_1, \dots, d_n)$, let $\mathscr{P}_{(d)} = \mathscr{P}_{d_1} \times \cdots \times \mathscr{P}_{d_n}$ so $f = (f_1, \dots, f_n) \in \mathscr{P}_{(d)}$ is a system of n polynomial equations in n complex variables and f_i has degree d_i.

As Smale's, Renegar's results were not finite average cost or time. In a series of papers Shub & Smale [1993a], Shub & Smale [1993b], Shub & Smale [1993c], Shub & Smale [1996], made some further changes and achieved enough results for Smale 17th problem to emerge a reasonable if challenging research goal. Let us recall the 17th problem from Smale [2000]:

Problem 17: Solving Polynomial Equations.

> *Can a zero of n-complex polynomial equations in n-unknowns be found approximately, on the average, in polynomial time with a uniform algorithm?*

In place of $\mathscr{P}_{(d)}$ and $\mathbb{C}^n$ it is natural to consider $\mathcal{H}_{(d)} = \mathcal{H}_{d_1} \times \cdots \times \mathcal{H}_{d_n}$ where $\mathcal{H}_{d_i}$ is the vector space of homogeneous polynomials of degree d_i in $n+1$ complex variables.

For $f \in \mathcal{H}_{(d)}$ and $\lambda \in \mathbb{C}$,

$$f(\lambda\zeta) = \Delta\left(\lambda^{d_i}\right) f(\zeta),$$

where $\Delta(a_i)$ means the diagonal matrix whose i-th diagonal entry is a_i. Thus the zeros of $f \in \mathcal{H}_{(d)}$ are now complex lines so may be considered as points in projective space $\mathbb{P}(\mathbb{C}^{n+1})$. The map

$$\mathrm{i}_{d_i} : \mathscr{P}_{d_i} \to \mathcal{H}_{d_i}, \qquad \mathrm{i}_{d_i}(f)(z_0, \ldots, z_n) = z_0^{d_i} f\left(\frac{z_1}{z_0}, \ldots, \frac{z_n}{z_0}\right),$$

is an isomorphism and $\mathrm{i} : \mathscr{P}_{(d)} \to \mathcal{H}_{(d)}$ for $\mathrm{i} = (\mathrm{i}_{d_1}, \ldots, \mathrm{i}_{d_n})$ is an isomorphism.

The affine chart

$$\mathrm{j} : \mathbb{C}^n \to \mathbb{P}(\mathbb{C}^{n+1}), \qquad \mathrm{j}(\zeta_1, \ldots, \zeta_n) = \mathbb{C}(1 : \zeta_1 : \ldots : \zeta_n),$$

maps the zeros of $f \in \mathscr{P}_{(d)}$ to zeros of $\mathrm{i}(f)$. In addition $\mathrm{i}(f)$ may have zeros at infinity i.e. zeros with $\zeta_0 = 0$.

From now on we consider $\mathcal{H}_{(d)}$ and $\mathbb{P}(\mathbb{C}^{n+1})$. On $\mathcal{H}_{d_i}$ we put a unitarily invariant Hermitian structure which we first encountered in the book Weyl [1939] and which is sometimes called Weyl, Bombieri-Weyl or Kostlan Hermitian structure depending on the applications considered.

For $\alpha = (\alpha_0, \dots, \alpha_n) \in \mathbb{N}^{n+1}$, $\|\alpha\| = d_i$ the monomial $x^\alpha = x_0^{\alpha_0} \cdots x_n^{\alpha_n}$, the Weyl Hermitian structure makes $\langle x^\alpha, x^\beta \rangle = 0$, for $\alpha \neq \beta$ and

$$\langle x^\alpha, x^\alpha \rangle = \binom{d_i}{\alpha}^{-1} = \left(\frac{d_i!}{\alpha_0! \cdots \alpha_n!} \right)^{-1}.$$

On $\mathcal{H}_{(d)}$ we put the product structure

$$\langle f, g \rangle = \sum_{i=1}^{n} \langle f_i, g_i \rangle.$$

On $\mathbb{C}^{n+1}$ we put the usual Hermitian structure

$$\langle x, y \rangle = \sum_{k=0}^{n} x_k \, \overline{y_k}.$$

Given a complex vector space V with Hermitian structure and a vector $0 \neq v \in V$, we let $v^\perp$ be the Hermitian complement of v,

$$v^\perp = \{ w \in V : \langle v, w \rangle = 0 \}.$$

$v^\perp$ is a model for the tangent space, $T_v\mathbb{P}(V)$, of the projective space $\mathbb{P}(V)$ at the equivalence class of v (which we also denote by v).

$T_v\mathbb{P}(V)$ inherits an Hermitian structure from $\langle \cdot, \cdot \rangle$ by the formula

$$\langle w_1, w_2 \rangle_v = \frac{\langle w_1, w_2 \rangle}{\langle v, v \rangle},$$

where w_1, $w_2 \in v^\perp$ represent the tangent vectors at $T_v\mathbb{P}(V)$.

This Hermitian structure which is well defined is called the Fubini-Study Hermitian structure.

The group of unitary transformations $\mathcal{U}(n+1)$ acts on $\mathcal{H}_{(d)}$ and $\mathbb{C}^{n+1}$ by $f \mapsto f \circ U^{-1}$ and $\zeta \mapsto U\zeta$ for $U \in \mathcal{U}(n+1)$.

This unitary action preserves the Hermitian structure on $\mathcal{H}_{(d)}$ and $\mathbb{C}^{n+1}$, see

Blum *et al.* [1998]. That is, for $U \in \mathcal{U}(n+1)$,

$$\langle f \circ U^{-1}, g \circ U^{-1} \rangle = \langle f, g \rangle \quad \text{for} \quad f,\, g \, \in \mathcal{H}_{(d)};$$
$$\langle U\zeta, U\zeta' \rangle = \langle \zeta, \zeta' \rangle \quad \text{for} \quad \zeta,\, \zeta' \, \in \mathbb{C}^{n+1}.$$

The zeros of λf and f for $0 \neq \lambda \in \mathbb{C}$ are the same, and we may consider the space $\mathbb{P}\left(\mathcal{H}_{(d)}\right)$. Now the space of problem instances is compact and the space $\mathbb{P}(\mathbb{C}^{n+1})$ is compact as well. $\mathbb{P}\left(\mathcal{H}_{(d)}\right)$ has a unitarily invariant Hermtitian structure which gives rise to a volume form of finite volume $\frac{\pi^{N-1}}{\Gamma(N)}$, where $N = \dim \mathcal{H}_{(d)}$.

The average of a function $\phi : \mathbb{P}\left(\mathcal{H}_{(d)}\right) \to \mathbb{R}$ is

$$\mathbb{E}(\phi) = \frac{1}{\mathrm{vol}(\mathbb{P}\left(\mathcal{H}_{(d)}\right))} \int_{f \in \mathbb{P}\left(\mathcal{H}_{(d)}\right)} \varphi(f)\, df = \frac{\Gamma(N)}{\pi^{N-1}} \int_{f \in \mathbb{P}\left(\mathcal{H}_{(d)}\right)} \varphi(f)\, df.$$

If ϕ is induced by a homogeneous function $\phi : \mathcal{H}_{(d)} \to \mathbb{R}$ of degree zero, that is, $\phi(\lambda f) = \phi(f)$, $\lambda \in \mathbb{C} - \{0\}$, then we may also compute this average with respect to the Gaussian measure on $(\mathcal{H}_{(d)}, \langle \cdot, \cdot \rangle)$, that is,

$$\mathbb{E}(\phi) = \frac{1}{(2\pi)^N} \cdot \int_{\mathcal{H}_{(d)}} \varphi(f) e^{-\|f\|^2/2}\, df.$$

And it is this approach via the Gaussians above defined on $\mathcal{H}_{(d)}$ and the Fubini-Study Hermitian structure and volume form on $\mathbb{P}(\mathbb{C}^{n+1})$ that we take in this chapter. The quantities we define on $\mathcal{H}_{(d)}$ are homogeneous of degree zero, thus are defined on $\mathbb{P}\left(\mathcal{H}_{(d)}\right)$ and benefit from the compactness of this space and of $\mathbb{P}(\mathbb{C}^{n+1})$. While averages over systems of equations may be carried out in the vector space $\mathcal{H}_{(d)}$.

The solution variety

$$\mathcal{V} = \{(f, x) \in (\mathcal{H}_{(d)} - \{0\}) \times \mathbb{P}(\mathbb{C}^{n+1}) :\ f(x) = 0\},$$

is a central object of study.

$\mathcal{V}$ is equipped with two projections:

$$\begin{tikzcd} & \mathcal{V} \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ \mathcal{H}_{(d)} & & \mathbb{P}(\mathbb{C}^{n+1}) \end{tikzcd}$$

The solution variety $\mathcal{V}$ also has a projective version, namely,

$$\mathcal{V}_{\mathbb{P}} = \{(f, x) \in \mathbb{P}\left(\mathcal{H}_{(d)}\right) \times \mathbb{P}(\mathbb{C}^{n+1}) : f(x) = 0\}.$$

3.1.1 Homotopy Methods

Homotopy methods for the solution of a system $f \in \mathcal{H}_{(d)}$ proceed as follows. Choose $(g, \zeta) \in \mathcal{V}$ a known pair. Connect g to f by a C^1 curve f_t in $\mathcal{H}_{(d)}$, $0 \leq t \leq 1$, such that $f_0 = g$, $f_1 = f$, and continue $\zeta_0 = \zeta$ to ζ_t such that $f_t(\zeta_t) = 0$, so that $f_1(\zeta_1) = 0$. By the implicit function theorem this continuation is possible for a generic set of C^1 paths in the C^1 topolgy, and indeed even for almost all "straight line" paths in $\mathcal{H}_{(d)}$, i.e. if ζ is a non-degenerate zero of g then for almost all f, ζ may be continued to a root of f along the curve $f_t = (1-t)g+tf$.

Now homotopy methods numerically approximate the path (f_t, ζ_t). One way to accomplish the approximation is via (projective) Newton's methods. Given an approximation x_t to ζ_t define

$$x_{t+\Delta t} = N_{f_{t+\Delta t}}(x_t),$$

where

$$N_f(x) = x - (Df(x)|_{x^\perp})^{-1} f(x).$$

Recall that x_t is an approximate zero of f_t associated with the zero ζ_t means that the sequence of Newton iteratives $N^k_{f_t}(x_t)$ converges immediately quadratically to ζ_t.

The main result of Shub [2009] is that Δt may be chosen so that $t_0 = 0$, $t_k = t_{k-1} + \Delta t_k$, x_{t_k} is an approximate zero of f_{t_k} with associated zero ζ_{t_k} and

$t_K = 1$ for

$$K = K(f, g, \zeta) \leq C\, D^{3/2} \int_0^1 \mu(f_t, \zeta_t)\, \|(\dot{f}_t, \dot{\zeta}_t)\|_{(f_t,\zeta_t)}\, dt. \tag{3.1.1}$$

Here C is a universal constant, $D = \max d_i$,

$$\mu(f, \zeta) = \|f\| \cdot \|(Df(\zeta)|_{\zeta^\perp})^{-1} \Delta(\|\zeta\|^{d_i - 1} \sqrt{d_i})\|$$

is the condition number of f at ζ, and

$$\|(\dot{f}_t, \dot{\zeta}_t)\|_{(f_t,\zeta_t)} = (\|\dot{f}_t\|_{f_t} + \|\dot{\zeta}_t\|_{\zeta_t})^{1/2}$$

is the norm of the tangent vector to the projected curve in (f_t, ζ_t) in $\mathcal{V}_{\mathbb{P}} \subset \mathbb{P}\left(\mathcal{H}_{(d)}\right) \times \mathbb{P}(\mathbb{C}^{n+1})$. The choice of Δt_k is made explicit in Dedieu *et al.* [2012].

In $\mathcal{V}_{\mathbb{P}}$, $\|\dot{\zeta}_t\|_{\zeta_t} \leq \mu(f_t, \zeta_t)\, \|\dot{f}_t\|_{f_t}$, so the estimates (3.1.1) may be bounded from above by

$$K(f, g, \zeta) \leq C\, D^{3/2} \int_0^1 \mu(f_t, \zeta_t)^2\, \|\dot{f}_t\|_{f_t}\, dt, \tag{3.1.2}$$

for a perhaps different universal constant C.

Finally in the case of straight line homotopy $\|\dot{f}_t\|_{f_t} = \dfrac{\sin(\theta)\, \|f_0\|\, \|f_1\|}{\|f_t\|^2}$, where θ is the angle between f_0 and f_1. So (3.1.2) may be rewritten as

$$K(f, g, \zeta) \leq C\, D^{3/2}\, \sin(\theta)\, \|f_0\|\, \|f_1\| \int_0^1 \frac{\mu(f_t, \zeta_t)^2}{\|f_t\|^2}\, dt, \tag{3.1.3}$$

see Bürgisser & Cucker [2011].

Much attention has been devoted to studying the right hand of (3.1.3), for a good starting point (g, ζ).

In Beltrán & Pardo [2009b], an affirmative probabilistic solution to Smale's 17th problem is proven. They prove that a random point (g, ζ) is good in the sense that with random fixed starting point $(g, \zeta) = (f_0, \zeta_0)$ the average value of the right hand side of (3.1.3) is bounded by $O(nN)$. Moreover, Beltrán and Pardo show how to pick a random starting point starting from a random $n \times (n + 1)$ matrix.

In [Bürgisser & Cucker, 2011] Bürgisser-Cucker produce a deterministic starting point with polynomial average cost, except for a narrow range of dimensions. More precisely:

> *There is a deterministic real number algorithm that on input $f \in \mathcal{H}_{(d)}$ computes an approximate zero of f in average time $N^{O(\log\log N)}$, where $N = \dim \mathcal{H}_{(d)}$ measures the size of the input f . Moreover, if we restrict data to polynomials satisfying*
>
> $$D \le n^{\frac{1}{1+\varepsilon}}, \quad \text{or} \quad D \ge n^{1+\varepsilon},$$
>
> *for some fixed $\varepsilon > 0$, then the average time of the algorithm is polynomial in the input size N.*

So Smale's 17th problem in its deterministic form remains open for a narrow range of degrees and variables.

3.1.2 Smale's Algorithm Reconsidered

Smale's 1981 algorithm depends on $f(0)$, so there is no fixed starting point for all systems. Given $\zeta \in \mathbb{P}(\mathbb{C}^{n+1})$ we define for $f \in \mathcal{H}_{(d)}$ the straight line segment $f_t \in \mathcal{H}_{(d)}$, $0 \le t \le 1$, by

$$f_t = f - (1-t)\Delta\left(\frac{\langle\cdot,\zeta\rangle^{d_i}}{\langle\zeta,\zeta\rangle^{d_i}}\right) f(\zeta).$$

So $f_0(\zeta) = 0$ and $f_1 = f$. Therefore we may apply homotopy methods to this line segment.

Note that if we restrict f to the affine chart $\zeta + \zeta^\perp$ then

$$f_t(z) = f(z) - (1-t)f(\zeta),$$

and if we take $\zeta = (1, 0, \dots, 0)$ and $n = 1$ we recover Smale's algorithm.

There is no reason to single out $\zeta = (1, 0, \dots, 0)$. Since the unitary group acts by isometries on $\mathbb{P}\left(\mathcal{H}_{(d)}\right)$, $\mathbb{P}(\mathbb{C}^{n+1})$, $\mathcal{V}$ and $\mathcal{V}_{\mathbb{P}}$, and preserves μ and is transitive on $\mathbb{P}(\mathbb{C}^{n+1})$, all the points ζ yield algorithms with the same average cost.

Note that if we let

$$\mathcal{V}_\zeta = \{f \in \mathcal{H}_{(d)} : f(\zeta) = 0\},$$

then

$$f_0 = f - \Delta\left(\frac{\langle \cdot, \zeta\rangle^{d_i}}{\langle \zeta, \zeta\rangle^{d_i}}\right) f(\zeta),$$

is the orthogonal projection $\Pi_\zeta(f)$ of f on $\mathcal{V}_\zeta$. This follows from the reproducing kernel property of the Weyl Hermitian product on $\mathcal{H}_{d_i}$, namely,

$$\langle g, \langle \cdot, \zeta\rangle^{d_i}\rangle = g(\zeta), \tag{3.1.4}$$

for all $g \in \mathcal{H}_{d_i}$, $(i = 1, \ldots, n)$. In particular $\|\langle \cdot, \zeta\rangle^{d_i}\| = \|\zeta\|^{d_i}$.

Then,

$$\|f - \Pi_\zeta(f)\| = \|\Delta(\|\zeta\|^{-d_i}) f(\zeta)\|,$$

while

$$\|\Pi_\zeta(f)\| = \left(\|f\|^2 - \|\Delta(\|\zeta\|^{-d_i}) f(\zeta)\|^2\right)^{1/2}.$$

Let $\Phi : \mathcal{H}_{(d)} \times \mathbb{P}(\mathbb{C}^{n+1}) \times [0,1] \to \mathcal{V}$ is the map given by

$$\Phi(f, \zeta, t) = (f_t, \zeta_t), \tag{3.1.5}$$

where

$$f_t = (1-t)\Pi_\zeta(f) + tf,$$

that is,

$$f_t = f - (1-t)\Delta\left(\frac{\langle \cdot, \zeta\rangle^{d_i}}{\langle \zeta, \zeta\rangle^{d_i}}\right) f(\zeta),$$

and ζ_t is the homotopy continuation of ζ along the path f_t.

Proposition 3.1.1. *For almost every $f \in \mathcal{H}_{(d)}$, the set of $\zeta \in \mathbb{P}(\mathbb{C}^{n+1})$ such that Φ is defined for all $t \in [0,1]$ has full measure. Moreover, for every $\zeta \in \mathbb{P}(\mathbb{C}^{n+1})$, the set of $f \in \mathcal{H}_{(d)}$ such that Φ is defined for all $t \in [0,1]$ has full measure.*

(See *Section 3.2* for a proof of *Proposition 3.1.1*).

Remark: In fact, the proof also shows that the complement of the set (f, ζ) such that Φ is defined for all $t \in [0,1]$ is a real algebraic set.

The norm of $\dot{f}_t$ is given now by the formula

$$\|\dot{f}_t\|_{f_t} = \frac{\|f_0\| \, \|f_1\| \, \sin(\theta)}{\|f_t\|^2} = \frac{\|\Pi_\zeta(f)\| \, \|f - \Pi_\zeta(f)\|}{\|f_t\|^2}$$
$$= \frac{\left(\|f\|^2 - \|\Delta(\|\zeta\|^{-d_i})f(\zeta)\|^2\right)^{1/2} \|\Delta(\|\zeta\|^{-d_i})f(\zeta)\|}{\|f_t\|^2}.$$

Let $K(f,\zeta) = K(f, \Pi_\zeta(f), \zeta)$ and $K_\zeta(f) = K(f,\zeta)$. Then, the average cost of this algorithm satisfy

Proposition 3.1.2.

$$\mathbb{E}(K_\zeta) = \mathbb{E}(K) \leq \textit{(I)},$$

where

$$\textit{(I)} = \frac{CD^{3/2}}{(2\pi)^N \, vol(\mathbb{P}(\mathbb{C}^{n+1}))} \cdot \int_{f \in \mathcal{H}_{(d)}} \int_{\zeta \in \mathbb{P}(\mathbb{C}^{n+1})} \int_{t \in [0,1]} \frac{\mu(f_t, \zeta_t)^2}{\|f_t\|^2} \cdot$$
$$\cdot \left(\|f\|^2 - \|\Delta(\|\zeta\|^{-d_i})f(\zeta)\|^2\right)^{1/2} \|\Delta(\|\zeta\|^{-d_i})f(\zeta)\| \, e^{-\|f\|^2/2} \, df \, d\zeta \, dt.$$

As we have mentioned above it is easy to see by unitary invariance of all the quantities involved that the average $\mathbb{E}(K_\zeta)$ is independent of ζ and equal to $\mathbb{E}(K)$. This argument proves the first equality of this proposition. The inequality follows immediately from the definition of $K(f,\zeta)$.

What is gained by letting ζ vary and dividing by $\mathrm{vol}(\mathbb{P}(\mathbb{C}^{n+1}))$ is a new way to see the integral which raises a collection of interesting questions.

Suppose η is a non-degenerate zero of $h \in \mathcal{H}_{(d)}$. We define the basin of η, $B(h,\eta)$, as those $\zeta \in \mathbb{P}(\mathbb{C}^{n+1})$ such that the zero ζ of $h - \Delta\left(\frac{\langle \cdot, \zeta\rangle^{d_i}}{\langle \zeta, \zeta\rangle^{d_i}}\right) h(\zeta)$ continues to η for the homotopy h_t. From the proof of *Proposition 3.1.1* we observe that the basins are open sets.

Let (I) be the expression defined on *Proposition 3.1.2.* Then, the main result of this chapter is

Theorem 5.

$$\textit{(I)} = \frac{CD^{3/2}\Gamma(n+1)2^{n-1}}{(2\pi)^N \pi^n} \int_{h \in \mathcal{H}_{(d)}} \Big[\sum_{\eta / \, h(\eta)=0} \frac{\mu^2(h,\eta)}{\|h\|^2} \Theta(h,\eta) \Big] e^{-\|h\|^2/2} \, dh,$$

where

$$\Theta(h,\eta) = \int_{\zeta\in B(h,\eta)} \frac{\left(\|h\|^2 - \|\Delta(\|\zeta\|^{-d_i})h(\zeta)\|^2\right)^{1/2}}{\|\Delta(\|\zeta\|^{-d_i})h(\zeta)\|^{2n-1}} \cdot \\ \cdot \Gamma(\|\Delta(\|\zeta\|^{-d_i})h(\zeta)\|^2/2, n) e^{\|\Delta(\|\zeta\|^{-d_i})h(\zeta)\|^2/2}\, d\zeta,$$

and $\Gamma(\alpha,n) = \int_\alpha^{+\infty} t^{n-1}e^{-t}\,dt$ *is the incomplete gamma function.*

Essentially nothing is known about the integrals.

(a) Is (I) finite for all or some n?

(b) Might (I) even be polynomial in N for some range of dimensions and degrees?

(c) What are the basins like? Even for $n = 1$ these are interesting questions.

The integral

$$\frac{1}{(2\pi)^N}\int_{h\in\mathcal{H}_{(d)}} \sum_{\eta/\,h(\eta)=0} \frac{\mu^2(h,\eta)}{\|h\|^2}\cdot e^{-\|h\|^2/2}\,dh \le \frac{e(n+1)}{2}\mathcal{D},$$

where $\mathcal{D} = d_1\cdots d_n$ is the Bézout number (see Bürgisser & Cucker [2011]). So the question is how does the factor $\Theta(h,\eta)$ affect the integral.

(d) Evaluate or estimate

$$\int_{\zeta\in\mathbb{P}(\mathbb{C}^{n+1})} \frac{1}{\|\Delta(\|\zeta\|^{-d_i})h(\zeta)\|^{2n-1}}\cdot e^{\frac{1}{2}\|\Delta(\|\zeta\|^{-d_i})h(\zeta)\|^2}\,d\zeta.$$

Note that

$$\|h\|_{L^p} = \left(\frac{1}{\mathrm{vol}(\mathbb{P}(\mathbb{C}^{n+1}))}\int_{\zeta\in\mathbb{P}(\mathbb{C}^{n+1})} \|\Delta(\|\zeta\|^{-d_i})h(\zeta)\|^p\,d\zeta\right)^{1/p},$$

for $p\ge 1$, is a different way to define a norm on h. For $p = 2$ we get another unitarily invariant Hermitian structure on $\mathcal{H}_{(d)}$, which differs from the Bombieri-Weyl by

$$\|h\|_{L^2}^2 = \sum_{i=1}^{n} \frac{d_i!n!}{(d_i+n)!}\|h_i\|^2,$$

(cf. [Dedieu, 2006, page 133])

If the integral in (d) can be controlled, if the integral on the $\mathcal{D}$ basins are reasonably balanced, the factor of $\mathcal{D}$ in (c) may be cancel.

Remark: The proof of *Theorem 5* involved complicated formulas which exhibited enormous calculations. We do not have a good explanation for this cancellation.

At the end of this chapter we present some numerical experiments with $n = 1$ and $d = 7$ which were done by Carlos Beltrán on the Altamira super computer at the Universidad de Cantabria (partially supported by MTM2010-16051 Spanish Ministry of Science and Innovation MICINN). It would be interesting to see more experimantal data. The proof of the *Theorem 5* is in *Section 3.3*.

3.2 Proof of Proposition 3.1.1

For the proof of *Proposition 3.1.1* we need a technical lemma.

Lemma 3.2.1. *Let E be a vector bundle over B, F be finite dimensional vector space, and consider the trivial vector bundle $F \times B$. Let $\varphi : F \times B \to E$ be a bundle map, covering the identity in B, which is a fiberwise surjective linear map. Then, φ is a surjective submersion.*

The proof is left to the reader.

Recall that $\Phi : \mathcal{H}_{(d)} \times \mathbb{P}(\mathbb{C}^{n+1}) \times [0,1] \to \mathcal{V}$ is the map given by

$$\Phi(f, \zeta, t) = (f_t, \zeta_t),$$

where

$$f_t = f - (1-t)\Delta\left(\frac{\langle \cdot, \zeta\rangle^{d_i}}{\langle \zeta, \zeta\rangle^{d_i}}\right) f(\zeta),$$

and ζ_t is the homotopy continuation of ζ along the path f_t.

This map is defined at (f,ζ,t) provided that $\mathrm{rank}(Df_t(\zeta_t)|_{\zeta_t^\perp}) = n$.

Let $\overline{K}$ be the vector bundle over $\mathbb{C}^{n+1} - \{0\}$ with fiber $\overline{K}_z = L(z^\perp, \mathbb{C}^n)$, $z \in \mathbb{C}^{n+1} - \{0\}$, where $L(z^\perp, \mathbb{C}^n)$ is the space of linear transformations from $z^\perp$ to $\mathbb{C}^n$. For $k = 0, \dots, n$, let $\overline{K_k}$ be the sub-bundle of rank k linear transformations. Note that $\overline{K_k}$ has $(n-k)^2$ complex codimension (c.f. Arnold *et al.* [1985]). These sub-bundles define a stratification of the bundle $\overline{K}$.

Lemma 3.2.2. *Let $\Omega^{(0)}$ be the set of pairs $(f,\zeta) \in \mathcal{H}_{(d)} \times \mathbb{P}(\mathbb{C}^{n+1})$ such that Φ is not defined for $t = 0$. Then $\Omega^{(0)}$ is a stratified set of smooth manifolds of complex codimension $(n-k)^2$, for $k = 0, \dots, n-1$.*

Proof. Let $\hat{\Omega}^{(0)}$ be the inverse image of $\Omega^{(0)}$ under the canonical projection $\mathcal{H}_{(d)} \times \mathbb{C}^{n+1} - \{0\} \to \mathcal{H}_{(d)} \times \mathbb{P}(\mathbb{C}^{n+1})$.

Let $\varphi : \mathcal{H}_{(d)} \times \mathbb{C}^{n+1} - \{0\} \to \overline{K}$ be the map $\varphi(f,\zeta) = Df(\zeta)|_{\zeta^\perp}$. For each $k = 0, \dots, n-1$, let $\hat{\Omega}^{(0)}_k = \varphi^{-1}(\overline{K_k})$. Since $Df_0(\zeta)|_{\zeta^\perp} = Df(\zeta)|_{\zeta^\perp}$, then $\hat{\Omega}^{(0)} = \cup_{k=0}^{n-1} \hat{\Omega}^{(0)}_k$.
Claim: φ is transversal to $\overline{K_k}$ for $k = 0, \dots, n-1$:
Note that $\varphi(f,\cdot) : \mathbb{C}^{n+1} - \{0\} \to \overline{K}$ is a section of the vector bundle $\overline{K}$ for each $f \in \mathcal{H}_{(d)}$. Moreover, for each $\zeta \in \mathbb{C}^{n+1} - \{0\}$, the linear map $\varphi(\cdot,\zeta) : \mathcal{H}_{(d)} \to \overline{K}_\zeta$ is a surjective linear map. This fact follows by construction: given $L \in \overline{K}_\zeta = L(\zeta^\perp, \mathbb{C}^n)$, let $\tilde{L} \in L(\mathbb{C}^{n+1}, \mathbb{C}^n)$ be any linear extension of L to $\mathbb{C}^{n+1}$. Then, the system $f = \Delta(\frac{\langle \cdot,\zeta\rangle^{d_i-1}}{\langle \zeta,\zeta\rangle^{d_i-1}})\tilde{L}(\cdot) \in \mathcal{H}_{(d)}$ satisfy $Df(\zeta)|_{\zeta^\perp} = L$. Then, the claim follows from *Lemma 3.2.1.*

Since φ is tranversal, we conclude that the inverse image of a stratification is a stratification of the same dimension (c.f. Arnold *et al.* [1985]). That is, $\hat{\Omega}^{(0)}$ is a stratification of complex submanifolds of complex codimension $(n-k)^2$, for $k = 0, \dots, n-1$.

Moreover, since each strata $\hat{\Omega}^{(0)}_k$ is transversal to the fiber of the canonical projection $\mathcal{H}_{(d)} \times \mathbb{C}^{n+1} - \{0\} \to \mathcal{H}_{(d)} \times \mathbb{P}(\mathbb{C}^{n+1})$, then, its image, $\Omega^{(0)}_k$, is a smooth manifold of codimension $(n-k)^2$, and the lemma follows. □

One can define the homotopy continuation of the pair $(f,\zeta) \in \mathcal{H}_{(d)} \times \mathbb{P}(\mathbb{C}^{n+1})$ for all $t \in [0,1]$ whenever $(f,\zeta) \notin \Omega^{(0)}$ and lies outside the subset of pairs such that there exist $(w,t) \in \mathbb{P}(\mathbb{C}^{n+1}) \times (0,1]$ satisfying the following equations:

$$f(w) = (1-t)\Delta\left(\frac{\langle w,\zeta\rangle^{d_i}}{\langle \zeta,\zeta\rangle^{d_i}}\right) f(\zeta), \quad \text{and} \quad \operatorname{rank}(Df_t(w)) < n.$$

Note that, since f_t is homogeneous, then $\operatorname{rank}(Df_t(w))$ is well defined on $w \in \mathbb{P}(\mathbb{C}^{n+1})$.

Differentiating f_t we get

$$Df_t(w) = Df(w) - (1-t)\Delta\left(\frac{d_i\langle w,\zeta\rangle^{d_i-1}\langle \cdot,\zeta\rangle}{\langle \zeta,\zeta\rangle^{d_i}}\right) f(\zeta).$$

Therefore, taking $s = 1-t$, we conclude that one can define the homotopy continuation of the pair $(f,\zeta) \in \mathcal{H}_{(d)} \times \mathbb{P}(\mathbb{C}^{n+1})$ for all $t \in [0,1]$ whenever $(f,\zeta) \notin \Omega^{(0)}$ and lies outside the subset of pairs such that there exist $(w,s) \in \mathbb{P}(\mathbb{C}^{n+1}) \times [0,1)$ satisfying, for some $k = 0,\ldots,n-1$, the following equations:

$$\Delta(\langle \zeta,\zeta\rangle^{d_i}) f(w) - s\,\Delta(\langle w,\zeta\rangle^{d_i}) f(\zeta) = 0, \tag{3.2.1}$$

$$\operatorname{rank}\left(\left[\Delta(\langle \zeta,\zeta\rangle^{d_i}) \cdot Df(w) - s\,\Delta(d_i\langle w,\zeta\rangle^{d_i-1}\langle \cdot,\zeta\rangle)\, f(\zeta)\right]\big|_{w^\perp}\right) = k. \tag{3.2.2}$$

Let $\Sigma' \subset \mathcal{V}$ be the set of critical points of the projection $\pi_1 : \mathcal{V} \to \mathcal{H}_{(d)}$, and let $\Sigma = \pi_1(\Sigma') \subset \mathcal{H}_{(d)}$ be the discriminant variety.

Note that if $f \in \Sigma$ then every $\zeta \in \mathbb{P}(\mathbb{C}^{n+1})$ satisfies equations (3.2.1) and (3.2.2) for $s = 0$ and $w \in \mathbb{P}(\mathbb{C}^{n+1})$ a critical root of f. Hence, it is natural to remove the discriminant variety Σ and the case $s = 0$ from this discussion.

Lemma 3.2.3. *Let $\Lambda \subset \mathcal{H}_{(d)} - \Sigma \times \mathbb{P}(\mathbb{C}^{n+1}) \times \mathbb{P}(\mathbb{C}^{n+1}) \times (0,1)$ be the set of tuples (f,ζ,w,s) such that equations (3.2.1) and (3.2.2) holds for some $k = 0,\ldots,n-1$. Then, Λ is stratified set of smooth manifolds of real codimension $2(n+(n-k)^2)$ for $k = 0,\ldots,n-1$.*

Proof. Similar to the preceding proof, for each $k = 0,\ldots,n-1$, we consider the set $\hat{\Lambda}_k \subset \mathcal{H}_{(d)} - \Sigma \times \mathbb{C}^{n+1} - \{0\} \times \mathbb{C}^{n+1} - \{0\} \times (0,1)$ of tuples (f,ζ,w,s) such that equations (3.2.1) and (3.2.2) holds .

Let $(f,\zeta,w,s) \in \hat{\Lambda}_k$ for some $k \in \{0,\ldots,n-1\}$. Since $f \notin \Sigma$ then $\langle w,\zeta\rangle \neq 0$.

Therefore from (3.2.1), equation (3.2.2) takes the form

$$\operatorname{rank}\left(\left(\langle w,\zeta\rangle Df(w)-\Delta(d_i)\, f(w)\, \langle\cdot,\zeta\rangle\right)|_{w^\perp}\right)=k,$$

for $k=0,\ldots,n-1$.

Let

$$F=(F_1,F_2):\mathcal{H}_{(d)}-\Sigma\times\mathbb{C}^{n+1}-\{0\}\times\mathbb{C}^{n+1}-\{0\}\times(0,1)\to\mathbb{C}^n\times\overline{K}$$

be the map defined by

$$\begin{aligned}F_1(f,\zeta,w,s)&=\Delta(\langle\zeta,\zeta\rangle^{d_i})f(w)-s\,\Delta(\langle w,\zeta\rangle^{d_i})f(\zeta)\in\mathbb{C}^n\\ F_2(f,\zeta,w,s)&=(w,(\langle w,\zeta\rangle Df(w)-\Delta(d_i)\, f(w)\,\langle\cdot,\zeta\rangle)\,|_{w^\perp})\in\overline{K}.\end{aligned}$$

Note that $\hat{\Lambda}_k=F^{-1}(\{0\}\times\overline{K_k})$.

Claim: F is transversal to $\{0\}\times\overline{K_k}$:

In fact, what we prove is that DF is surjective at any point (f,ζ,w,s) which maps into $\{0\}\times\overline{K_k}$, for any $k=0,\ldots,n-1$, that is, any point in $\hat{\Lambda}_k$.

Recall that $\mathcal{V}_\zeta=\{f\in\mathcal{H}_{(d)}:\ f(\zeta)=0\}$. Consider the following orthogonal decomposition $\mathcal{H}_{(d)}=\mathcal{V}_\zeta\oplus C_\zeta$, where $C_\zeta={\mathcal{V}_\zeta}^\perp$.

Let $(f,\zeta,w,s)\in\hat{\Lambda}_k$. We first prove that $DF_1(f,\zeta,w,s)|_{C_\zeta}:C_\zeta\to\mathbb{C}^n$ is surjective.

Note that the linear map $\xi:\mathbb{C}^n\to C_\zeta$ given by $\xi(a)=\Delta\big(\frac{\langle\cdot,\zeta\rangle^{d_i}}{\langle\zeta,\zeta\rangle^{d_i}}\big)a$, is an isomorphism, where $\xi^{-1}:C_\zeta\to\mathbb{C}^n$ is given by $\xi^{-1}(f)=f(\zeta)$. Then, under this identification, the restriction to C_ζ of the derivative of F_1 is the linear map given by

$$DF_1(f,\zeta,w,s)\big|_{C_\zeta}=(1-s)\Delta(\langle w,\zeta\rangle^{d_i}),$$

for all tuples (f,ζ,w,s). Moreover, since $(f,\zeta,w,s)\in\hat{\Lambda}_k$, then $\langle w,\zeta\rangle\neq 0$ and $s\neq 1$, hence $DF_1(f,\zeta,w,s)\big|_{C_\zeta}$ is onto.

Now we prove that $DF_2(f,\zeta,w,s)\big|_{\mathcal{V}_\zeta\times T_w\mathbb{P}(\mathbb{C}^{n+1})}$ is surjective onto the tangent space $T_{F_2(f,\zeta,w,s)}\overline{K}$, at every $(f,\zeta,w,s)\in\hat{\Lambda}_k$.

Note that the map $F_2(f,\zeta,\cdot,s):\mathbb{C}^{n+1}-\{0\}\to\overline{K}$ is a section of the vector bundle $\overline{K}$. Therefore, from *Lemma 3.2.1*, it is enough to prove that $F_2|_{\mathcal{H}_{(d)}}(\cdot,\zeta,w,s)$ is a fiberwised surjective linear map.

Fix a tuple $(f, \zeta, w, s) \in \hat{\Lambda}_k$, for some $k = 0, \ldots, n-1$. The unitary group $\mathcal{U}(n+1)$ acts by isometries on $\mathcal{H}_{(d)} - \Sigma \times \mathbb{C}^{n+1} - \{0\} \times \mathbb{C}^{n+1} - \{0\}$ by $U \cdot (f, \zeta, w) = (f \circ U^{-1}, U(\zeta), U(w))$, and leave $\hat{\Lambda}_k$ invariant. Therefore we may assume that $w = e_0$. Write $f_i(z) = \sum_{\|\alpha\|=d_i} a_\alpha^{(i)} z^\alpha$, $(i = 1, \ldots, n)$. Then, the linear map $F_2(\cdot, \zeta, e_0, s) : \mathcal{H}_{(d)} \to \overline{K}_{e_0}$ is given by

$$F_2(f, \zeta, e_0, s) = ((\overline{\zeta_0}\, a^{(i)}_{(d_i-1, v_j)} - d_i a^{(i)}_{(d_i, 0, \ldots, 0)} \overline{\zeta_j}))_{i,j=1,\ldots,n},$$

where v_j is the n-vector with the j-entry equal to 1 and the others entries equal to 0.

In particular, since $\zeta_0 \neq 0$, the restriction $F_2(\cdot, \zeta, e_0, s) : \mathcal{V}_\zeta \to \overline{K}_{e_0}$ is surjective, concluding the claim.

Then, since F is tranversal to $\{0\} \times \overline{K_k}$, we conclude that $\hat{\Lambda}_k = F^{-1}(\{0\} \times \overline{K_k})$ is a submanifold of real codimension $2(n + (n-k)^2)$, for $k = 0, \ldots, n-1$.

To end the proof, we note that $\hat{\Lambda}_k$ is transversal to the fiber of the canonical projection $\mathcal{H}_{(d)} - \Sigma \times \mathbb{C}^{n+1} - \{0\} \times \mathbb{C}^{n+1} - \{0\} \times (0,1) \to \mathcal{H}_{(d)} - \Sigma \times \mathbb{P}(\mathbb{C}^{n+1}) \times \mathbb{P}(\mathbb{C}^{n+1}) \times (0,1)$. $\square$

Let $\Pi : \mathcal{H}_{(d)} \times \mathbb{P}(\mathbb{C}^{n+1}) \times \mathbb{P}(\mathbb{C}^{n+1}) \times (0,1) \to \mathcal{H}_{(d)} \times \mathbb{P}(\mathbb{C}^{n+1})$ be the canonical projection

$$\Pi(f, \zeta, w, s) = (f, \zeta).$$

Then, from *Lemma 3.2.2* and *Lemma 3.2.3* the set of pairs $(f, \zeta) \in \mathcal{H}_{(d)} \times \mathbb{P}(\mathbb{C}^{n+1})$ such that the homotopy is not defined for all $t \in [0,1]$ is contained by the union

$$\Omega^{(0)} \cup \Pi(\Lambda) \cup \Sigma \times \mathbb{P}(\mathbb{C}^{n+1}) \subset \mathcal{H}_{(d)} \times \mathbb{P}(\mathbb{C}^{n+1}).$$

Remark: We could conclude the proof by Fubini's Theorem. But we give a different argument. See the remark at the end.

Proof of Proposition 3.1.1. For $k = 0, \ldots, n-1$, let $\Omega_k^{(0)} \subset \mathcal{H}_{(d)} \times \mathbb{P}(\mathbb{C}^{n+1})$ be the subset given in the proof of *Lemma 3.2.2*, and let $\hat{\pi}_1 : \mathcal{H}_{(d)} \times \mathbb{P}(\mathbb{C}^{n+1}) \to \mathcal{H}_{(d)}$ be the projection in the first coordinate. From Sard's Lemma we get that almost every $f \in \mathcal{H}_{(d)}$ is a regular value of the restriction $\hat{\pi}_1|_{\Omega_k^{(0)}} : \Omega_k^{(0)} \to \mathcal{H}_{(d)}$, for each $k = 0, \ldots, n-1$. Therefore, from *Lemma 3.2.2*, we conclude that for almost every

$f \in \mathcal{H}_{(d)}$ the subset

$$\hat{\pi}_1|_{\Omega_k^{(0)}}{}^{-1}(f) = \hat{\pi}_1^{-1}(f) \cap \Omega_k^{(0)} \subset \mathbb{P}(\mathbb{C}^{n+1}),$$

is an empty set or a smooth submanifold of complex dimension $n - (n-k)^2$, for $k = 0, \ldots, n-1$. Hence, for almost every $f \in \mathcal{H}_{(d)}$, the set of $\zeta \in \mathbb{P}(\mathbb{C}^{n+1})$ such that Φ is not defined at $t = 0$ has measure zero.

Similar to the preceding argument, for each $k = 0, \ldots, n-1$, let $\Lambda_k \subset \mathcal{H}_{(d)} - \Sigma \times \mathbb{P}(\mathbb{C}^{n+1}) \times \mathbb{P}(\mathbb{C}^{n+1}) \times \mathbb{P}(\mathbb{C}^{n+1}) \times (0,1)$ be the set of tuples (f, ζ, w, s) such that equations (3.2.1) and (3.2.2) holds, and let $\hat{\Pi}_f : \mathcal{H}_{(d)} \times \mathbb{P}(\mathbb{C}^{n+1}) \times \mathbb{P}(\mathbb{C}^{n+1}) \times \mathbb{P}(\mathbb{C}^{n+1}) \times (0,1) \to \mathcal{H}_{(d)}$ be the projection in the first coordinate. Then by Sard's Lemma, almost every $f \in \mathcal{H}_{(d)}$ is a regular value of the restriction $\hat{\Pi}_f|_{\Lambda_k} : \Lambda_k \to \mathcal{H}_{(d)}$. Therefore, from *Lemma 3.2.3*, we conclude that for almost every $f \in \mathcal{H}_{(d)}$ the subset

$$\hat{\Pi}_f|_{\Lambda_k}{}^{-1}(f) = \hat{\Pi}_f^{-1}(f) \cap \Lambda_k \subset \mathbb{P}(\mathbb{C}^{n+1}) \times \mathbb{P}(\mathbb{C}^{n+1}) \times (0,1),$$

is an empty set or a smooth submanifold of real dimension $2n+1-2(n-k)^2$, for $k = 0, \ldots, n-1$. Then, projecting in the ζ-space we obtain that for almost every $f \in \mathcal{H}_{(d)}$, the set of $\zeta \in \mathbb{P}(\mathbb{C}^{n+1})$ such that Φ is not defined at $t \in (0,1)$ is a finite union of measure zero sets. The proof of the first statement of the proposition follows.

The second statement of *Propostion 3.1.1* follows directly from proofs of the claims of *Lemma 3.2.2*, and*Lemma 3.2.3*, and the subsecuent analysis of dimensions. $\square$

Remark: The proof of *Propostion 3.1.1* follows immediately from Fubini's Theorem. But we say more because this discussion may be useful for the discussion of the basins (recall question (c) after the statement of the main theorem). This proposition proves that the boundary of the basins are contained in this stratified set, the structure of which should be persistent by the isotopy theorem (c.f. Arnold *et al.* [1985]) on the connected components of the complement of the critical values of the projection. We don't know if there is more than one component.

3.3 Proof of Theorem 5

Let us first state the notation in the forthcoming computations. Most of the maps are defined between Hermitian spaces, however they are real differentiable. Therefore, unless we mention the contrary, all derivatives are real derivatives. Moreover, if a map is defined on $\mathbb{P}(\mathbb{C}^{n+1})$ then is natural to restrict its derivative at ζ to the complex tangent space $T_\zeta\mathbb{P}(\mathbb{C}^{n+1})$. If $L : E \to F$ is a linear map between finite dimensional Hermitian vector spaces, then its determinant, $\det(L)$, is the determinant of the linear map $L : E \to \mathrm{Im}(L)$, computed with respect to the associated canonical real structures, namely, the real part of the Hermitian product of E and the real part of the inherted Hermitian product on $\mathrm{Im}(L) \subset F$. The adjoint operator $L^* : F \to E$ is the is also computed with respect to the associated canonical real structures.

In general, if E is a set, Id_E means the identity map defined on that set.

Since the set of triples $(f, \zeta, t) \in \mathcal{H}_{(d)} \times \mathbb{P}(\mathbb{C}^{n+1}) \times [0, 1]$ such that that $t = 0$ or $t = 1$ has measure zero, we may assume in the rest of this section that $t \in (0, 1)$.

Recall that $\Phi : \mathcal{H}_{(d)} \times \mathbb{P}(\mathbb{C}^{n+1}) \times [0, 1] \to \mathcal{V}$ is the map given by

$$\Phi(f, \zeta, t) = (f_t, \zeta_t),$$

where

$$f_t = f - (1-t)\Delta\left(\frac{\langle\cdot, \zeta\rangle^{d_i}}{\langle\zeta, \zeta\rangle^{d_i}}\right) f(\zeta),$$

and ζ_t is the homotopy continuation of ζ along the path f_t.

For each $t \in (0, 1)$, let $\Phi_t : \mathcal{H}_{(d)} \times \mathbb{P}(\mathbb{C}^{n+1}) \to \mathcal{V}$ be the restriction $\Phi_t(\cdot, \cdot) = \Phi(\cdot, \cdot, t)$.

Recall that for each non-degenerate root η of h, $B(h, \eta)$ is the non-empty open set of those $\zeta \in \mathbb{P}(\mathbb{C}^{n+1})$ such that the zero ζ of $\Pi_\zeta(h)$ continues to η for the homotopy $h_t = (1-t)\Pi_\zeta(h) + th$.

Lemma 3.3.1. *Let $t \in (0, 1)$, and let $(h, \eta) \in \mathcal{V}$ be a regular value of Φ_t. Then, the fiber $\Phi_t(h, \eta)^{-1}$ is given by*

$$\Phi_t^{-1}(h, \eta) = \hat{H}_t(B(h, \eta)),$$

where $\hat{H}_t = (\hat{h}_t, Id_{\mathbb{P}(\mathbb{C}^{n+1})}) : \mathbb{P}(\mathbb{C}^{n+1}) \to \mathcal{H}_{(d)} \times \mathbb{P}(\mathbb{C}^{n+1})$ *and*

$$\hat{h}_t(\zeta) = h + \left(\frac{1-t}{t}\right) \Delta \left(\frac{\langle \cdot, \zeta\rangle^{d_i}}{\langle \zeta, \zeta\rangle^{d_i}}\right) h(\zeta). \tag{3.3.1}$$

Proof. For $0 < t < 1$, we have that $(f, \zeta) \in \Phi_t^{-1}(h, \eta)$ provided that

i) $h = f_t = tf + (1-t)\Pi_\zeta(f)$;

ii) the homotopy continuation of ζ on the path $\{sh + (1-s)\Pi_\zeta(f)\}_{s\in[0,1]}$ is η.

Since $\Pi_\zeta(h) = \Pi_\zeta(f)$ we conclude that

$$f = \frac{1}{t}\left(h - (1-t)\Pi_\zeta(h)\right) = h + \left(\frac{1-t}{t}\right)(h - \Pi_\zeta(h)),$$

and $\zeta \in B(h, \eta)$. □

Proposition 3.3.1. *Let* $(f, \zeta) \in \mathcal{H}_{(d)} \times \mathbb{P}(\mathbb{C}^{n+1})$ *such that* Φ_t *is defined and let* $(h, \eta) = \Phi_t(f, \zeta)$. *Then the normal jacobian of* Φ_t *is given by*

$$NJ_{\Phi_t}(f, \zeta) = t^{2n} \frac{Jac_{\hat{H}_t}(\zeta)}{NJ_{\pi_1}(h, \eta)},$$

where $Jac_{\hat{H}_t}(\zeta) = |\det(D\hat{H}_t(\zeta))|$ *is the jacobian of the map* $\hat{H}_t$ *defined in Lemma 3.3.1.*

The proof of this proposition is divided in several lemmas and is left to the end.

Proof of Theorem 5. Recall from *Proposition 3.1.2* that (I) is defined by

$$\begin{aligned}(\mathrm{I}) = &\frac{CD^{3/2}}{(2\pi)^N \operatorname{vol}(\mathbb{P}(\mathbb{C}^{n+1}))} \cdot \int_{f\in\mathcal{H}_{(d)}} \int_{\zeta\in\mathbb{P}(\mathbb{C}^{n+1})} \int_{t\in[0,1]} \frac{\mu(f_t, \zeta_t)^2}{\|f_t\|^2} \cdot \\ &\cdot \|\Pi_\zeta(f)\| \, \|\Delta(\|\zeta\|^{-d_i}) f(\zeta)\| \, e^{-\|f\|^2/2} \, df \, d\zeta \, dt.\end{aligned}$$

Then, for $0 < t < 1$, by the co-area formula for the map $\Phi_t : \mathcal{H}_{(d)} \times \mathbb{P}(\mathbb{C}^{n+1}) \to \mathcal{V}$, and *Proposition 3.3.1* we obtain

$$(\mathrm{I}) = \frac{CD^{3/2}}{(2\pi)^N \operatorname{vol}(\mathbb{P}(\mathbb{C}^{n+1}))} \cdot \int_0^1 t^{-2n} \int_{(h,\eta)\in\mathcal{V}} \frac{\mu(h,\eta)^2}{\|h\|^2} NJ_{\pi_1}(h,\eta) \cdot$$
$$\cdot \int_{(f,\zeta)\in\Phi_t^{-1}(h,\eta)} \frac{\|\Pi_\zeta(f)\| \, \|\Delta(\|\zeta\|^{-d_i}) f(\zeta)\|}{Jac_{\hat{H}_t}(\zeta)} e^{-\|f\|^2/2} \, dt \, d\mathcal{V} \, d\Phi_t^{-1}(h,\eta).$$

If $\Phi_t(f,\zeta) = (h,\zeta)$ then $f(\zeta) = h(\zeta)/t$, $\Pi_\zeta(f) = \Pi_\zeta(h)$. From *Lemma 3.3.1* we obtain that, for all $t \in (0,1)$, $\hat{H}_t : B(h,\eta) \to \Phi_t^{-1}(h,\eta)$ given by $\zeta \mapsto (\hat{h}_t(\zeta), \zeta)$, is a parametrization of the fiber $\Phi_t^{-1}(h,\eta)$. Moreover, since $\zeta = \hat{H}_t^{-1}(f,\zeta))$ whenever $\hat{H}_t(\zeta) = (f,\zeta)$, then applying the change of variable formula we conclude that

$$(\mathrm{I}) = \frac{CD^{3/2}}{(2\pi)^N \operatorname{vol}(\mathbb{P}(\mathbb{C}^{n+1}))} \cdot \int_0^1 t^{-2n-1} \int_{(h,\eta)\in\mathcal{V}} \frac{\mu(h,\eta)^2}{\|h\|^2} NJ_{\pi_1}(h,\eta) \cdot \tag{3.3.2}$$
$$\cdot \int_{\zeta\in B(h,\eta)} \|\Pi_\zeta(h)\| \, \|\Delta(\|\zeta\|^{-d_i}) h(\zeta)\| \, e^{-\|\hat{h}_t(\zeta)\|^2/2} \, dt \, d\mathcal{V} \, d\zeta.$$

From the definition of $\hat{h}_t(\zeta)$ in (3.3.1) and the reproducing kernel property of the Weyl Hermitian product (3.1.4), we obtain

$$\|\hat{h}_t(\zeta)\|^2 = \|h\|^2 + 2\left(\frac{1-t}{t}\right) \operatorname{Re}\langle h, \Delta(\langle\zeta,\zeta\rangle^{-d_i}\langle\cdot,\zeta\rangle^{d_i}) h(\zeta)\rangle +$$
$$+ \left(\frac{1-t}{t}\right)^2 \|\Delta(\langle\zeta,\zeta\rangle^{-d_i}\langle\cdot,\zeta\rangle^{d_i}) h(\zeta)\|^2,$$

then

$$\|\hat{h}_t(\zeta)\|^2 = \|h\|^2 - \left(1 - \frac{1}{t^2}\right) \|\Delta(\|\zeta\|^{-d_i}) h(\zeta)\|^2. \tag{3.3.3}$$

From the change of variable $u = \alpha^2/(2t^2)$, one gets that

$$\int_0^1 \frac{1}{t^{2n+1}} e^{-\alpha^2/(2t^2)} \, dt = \frac{2^{n-1}}{\alpha^{2n}} \int_{\alpha^2/2}^{+\infty} u^{n-1} e^{-u} \, du, \tag{3.3.4}$$

where the last integral is the incomplete gamma function $\Gamma(\alpha^2/2, n)$. Then, from (3.3.2), (3.3.3), (3.3.4), and the fact that $\operatorname{vol}(\mathbb{P}(\mathbb{C}^{n+1})) = \pi^n/\Gamma(n+1)$ we obtain

$$(\mathrm{I}) = \frac{CD^{3/2}\Gamma(n+1)2^{n-1}}{(2\pi)^N \pi^n} \int_{(h,\eta)\in\mathcal{V}} \frac{\mu(h,\eta)^2}{\|h\|^2} NJ_{\pi_1}(h,\eta) \cdot \Theta(h,\eta) \, d\mathcal{V},$$

where

$$\Theta(h,\eta)=\int_{\zeta\in B(h,\eta)}\frac{\left(\|h\|^2-\|\Delta(\|\zeta\|^{-d_i})h(\zeta)\|^2\right)^{1/2}}{\|\Delta(\|\zeta\|^{-d_i})h(\zeta)\|^{2n-1}}\cdot$$
$$\cdot\,\Gamma(\|\Delta(\|\zeta\|^{-d_i})h(\zeta)\|^2/2,n)e^{\|\Delta(\|\zeta\|^{-d_i})h(\zeta)\|^2/2}\,d\zeta.$$

Now, the proof of *Theorem 5* follows applying the co-area formula for the projection $\pi_1:\mathcal{V}\to\mathcal{H}_{(d)}$. □

3.3.1 Proof of Proposition 3.3.1

The map $\hat{h}_t:\mathbb{P}(\mathbb{C}^{n+1})\to\mathcal{H}_{(d)}$ given in (3.3.1) is differentiable, and therefore $\hat{H}_t$ is also differentiable.

Lemma 3.3.2. *Let $(f,\zeta)\in\mathcal{H}_{(d)}\times\mathbb{P}(\mathbb{C}^{n+1})$ such that Φ_t is defined and let $(h,\eta)=\Phi_t(f,\zeta)$. Then,*

$$NJ_{\Phi_t}(f,\zeta)=\frac{\left|\det\left[D(\pi_1\circ\Phi_t)(\hat{h}_t(\zeta),\zeta)\cdot\left(Id_{\mathcal{H}_{(d)}},-(D\hat{h}_t(\zeta)|_{\zeta^\perp})^*\right)\right]\right|}{\left|\det(Id_{\zeta^\perp}+(D\hat{h}_t(\zeta)|_{\zeta^\perp})^*\cdot D\hat{h}_t(\zeta))|_{\zeta^\perp}\right|^{1/2}\cdot NJ_{\pi_1}(h,\eta)},$$

where $\left(Id_{\mathcal{H}_{(d)}},-(D\hat{h}_t(\zeta)|_{\zeta^\perp})^*\right):\mathcal{H}_{(d)}\to\mathcal{H}_{(d)}\times T_\zeta\mathbb{P}(\mathbb{C}^{n+1})$ is the linear map $\dot{f}\mapsto(\dot{f},-(D\hat{h}_t(\zeta)|_{\zeta^\perp})^*\dot{f})$.

Proof. In general, let E_1 and E_2 be finite dimensional vector spaces with inner product. Let $V\subset E_1\times E_2$ be a vector subspace such that $\dim(V)=\dim(E_1)$, and consider on V the inherited inner product. Let $\gamma:E_2\to E_1$ and $\alpha:E_1\times E_2\to V$ be linear operators. Consider the following diagram:

$$\begin{tikzcd}
 & E_1\times E_2 \arrow[r, "\alpha"] & V \arrow[d, "\pi_1"] \\
E_1 \arrow[ur, "(Id_{E_1},-\gamma^*)"] & E_2 \arrow[u, "(\gamma, Id_{E_2})"'] & E_1
\end{tikzcd}$$

where $(\gamma,Id_{E_2}):E_2\to E_1\times E_2$, and $\pi:V\to E_1$ is the restriction of the canonical projection in the first coordinate.

Note that the image of the operator $(Id_{E_1}, -\gamma^*) : E_1 \to E_1 \times E_2$ is the orthogonal complement of $(\gamma, Id)(E_2)$ in $E_1 \times E_2$, therefore, assuming that π_1 is an isomorphism, we get,

$$\begin{aligned} |\det(\alpha|_{((\gamma, Id_{E_2})(E_2))^\perp})| &= \frac{|\det(\pi_1 \cdot \alpha \cdot (Id_{E_1}, -\gamma^*))|}{|\det(\mathrm{Id}_{E_1} + \gamma \cdot \gamma^*)|^{1/2} \cdot |\det(\pi_1)|} \\ &= \frac{|\det(\pi_1 \cdot \alpha \cdot (Id_{E_1}, -\gamma^*))|}{|\det(\mathrm{Id}_{E_2} + \gamma^* \cdot \gamma)|^{1/2} \cdot |\det(\pi_1)|}, \end{aligned}$$

where the last equality follows by Sylvester Theorem: if A and B are matrices of size $n \times m$ and $m \times n$ respectively, then

$$\det(\mathrm{Id}_m + BA) = \det(\mathrm{Id}_n + AB). \tag{3.3.5}$$

Now the proof follows taking $E_1 = \mathcal{H}_{(d)}$, $E_2 = T_\zeta\mathbb{P}(\mathbb{C}^{n+1})$, $V = \mathcal{V}$, with the associated real inner products, $\gamma = D\hat{h}_t(\zeta)|_{\zeta^\perp}$ and $\alpha = D\Phi_t(\hat{h}_t, \zeta)|_{\mathcal{H}_{(d)} \times \zeta^\perp}$. $\square$

The derivative of $\hat{h}_t$ at $\zeta \in \mathbb{P}(\mathbb{C}^{n+1})$ in the direction $\dot{\zeta} \in T_\zeta\mathbb{P}(\mathbb{C}^{n+1})$ is given by

$$D\hat{h}_t(\zeta)\dot{\zeta} = \left(\frac{1-t}{t}\right) \cdot (K_\zeta(\dot{\zeta}) + L_\zeta(\dot{\zeta})),$$

where K_ζ, $L_\zeta : T_\zeta\mathbb{P}(\mathbb{C}^{n+1}) \to \mathcal{H}_{(d)}$ are given by

$$K_\zeta(\dot{\zeta}) = \Delta\left(\frac{\langle \cdot, \zeta\rangle^{d_i}}{\langle \zeta, \zeta\rangle^{d_i}}\right) \cdot Dh(\zeta)\dot{\zeta}; \tag{3.3.6}$$

$$L_\zeta(\dot{\zeta}) = \Delta\left(\frac{d_i\langle \cdot, \zeta\rangle^{d_i-1}\langle \cdot, \dot{\zeta}\rangle}{\langle \zeta, \zeta\rangle^{d_i}}\right) h(\zeta), \tag{3.3.7}$$

for all $\dot{\zeta} \in T_\zeta\mathbb{P}(\mathbb{C}^{n+1})$.

Lemma 3.3.3. *The adjoints operators $K_\zeta{}^*$, $L_\zeta{}^* : \mathcal{H}_{(d)} \to T_\zeta\mathbb{P}(\mathbb{C}^{n+1})$, are given by*

$$K_\zeta{}^*(\dot{f}) = (Dh(\zeta)|_{\zeta^\perp})^* \cdot \Delta(\langle \zeta, \zeta\rangle^{-d_i+1})\dot{f}(\zeta), \tag{3.3.8}$$

and

$$L_\zeta{}^*(\dot{f}) = (D\dot{f}(\zeta)|_{\zeta^\perp})^* \cdot \Delta(\langle \zeta, \zeta\rangle^{-d_i+1})h(\zeta), \tag{3.3.9}$$

for any $\dot{f} \in \mathcal{H}_{(d)}$.

Proof. By the definition of adjoint, the definition of K_ζ and the reproducing kernel property of the Weyl Hermitian product (3.1.4), we get

$$\begin{aligned}
\operatorname{Re}\langle {K_\zeta}^*(\dot f), \dot\zeta\rangle &= \|\zeta\|^2 \operatorname{Re}\langle \dot f, \Delta\big(\langle\zeta,\zeta\rangle^{-d_i}\langle\cdot,\zeta\rangle^{d_i}\big)\cdot Dh(\zeta)\dot\zeta\rangle \\
&= \operatorname{Re}\langle \dot f(\zeta), \Delta\big(\langle\zeta,\zeta\rangle^{-d_i+1}\big)\cdot Dh(\zeta)\dot\zeta\rangle \\
&= \operatorname{Re}\langle (Dh(\zeta)|_{\zeta^\perp})^*\cdot\Delta(\langle\zeta,\zeta\rangle^{-d_i+1})\dot f(\zeta), \dot\zeta\rangle.
\end{aligned}$$

Moreover, differentiating equation (3.1.4) with respect to ζ, we obtain for ${L_\zeta}^*$ that

$$\begin{aligned}
\operatorname{Re}\langle {L_\zeta}^*(\dot f), \dot\zeta\rangle &= \|\zeta\|^2 \operatorname{Re}\langle \dot f, \Delta(\langle\zeta,\zeta\rangle^{-d_i} d_i\langle\cdot,\zeta\rangle^{d_i-1}\langle\cdot,\dot\zeta\rangle)h(\zeta)\rangle \\
&= \operatorname{Re}\langle D\dot f(\zeta)\dot\zeta, \Delta(\langle\zeta,\zeta\rangle^{-d_i+1})h(\zeta)\rangle \\
&= \operatorname{Re}\langle (D\dot f(\zeta)|_{\zeta^\perp})^*\cdot\Delta(\langle\zeta,\zeta\rangle^{-d_i+1})h(\zeta), \dot\zeta\rangle.
\end{aligned}$$

□

Lemma 3.3.4. *One has,*

$$\begin{aligned}
&\left|\det(Id_{\zeta^\perp} + (D\hat h_t(\zeta)|_{\zeta^\perp})^*\cdot D\hat h_t(\zeta)|_{\zeta^\perp})\right| = \\
&\left(1+\left(\frac{1-t}{t}\right)^2 \|\Delta(\sqrt{d_i}\|\zeta\|^{-d_i})h(\zeta)\|^2\right)^{2n}\cdot \\
&\qquad\left|\det\left(Id_{\zeta^\perp} + \frac{\left(\frac{1-t}{t}\right)^2 (Dh(\zeta)|_{\zeta^\perp})^*\cdot\Delta\left(\|\zeta\|^{-d_i+1}\right)^2\cdot Dh(\zeta)_{\zeta^\perp}}{1+\left(\frac{1-t}{t}\right)^2\left\|\Delta\left(\sqrt{d_i}\|\zeta\|^{-d_i}\right)h(\zeta)\right\|^2}\right)\right|.
\end{aligned}$$

Proof. By direct computation we get

$$ {K_\zeta}^*\cdot K_\zeta = (Dh(\zeta)|_{\zeta^\perp})^*\cdot\Delta(\langle\zeta,\zeta\rangle^{-d_i+1})\cdot Dh(\zeta)|_{\zeta^\perp}; $$

$$ {K_\zeta}^*\cdot L_\zeta = {L_\zeta}^*\cdot K_\zeta = 0. $$

Note that, if $\dot f = L_\zeta(\dot\zeta)$ for some $\dot\zeta \in T_\zeta\mathbb{P}(\mathbb{C}^{n+1})$, then, for all $\theta\in\mathbb{C}^n$ we get

$$ (D\dot f(\zeta)|_{\zeta^\perp})^*\theta = \left(\operatorname{Re}\langle\theta, \Delta\left(\frac{d_i}{\|\zeta\|^2}\right)h(\zeta)\rangle\right)\dot\zeta. $$

Hence,

$$L_\zeta{}^* L_\zeta = \left\| \Delta\left(\sqrt{d_i}\|\zeta\|^{-d_i}\right) h(\zeta)\right\|^2 \cdot \mathrm{Id}_{\zeta^\perp}.$$

Therefore we get:

$$(D\hat{h}_t(\zeta)|_{\zeta^\perp})^* \cdot D\hat{h}_t(\zeta)|_{\zeta^\perp} = \left(\frac{1-t}{t}\right)^2 \left(K_\zeta{}^* \cdot K_\zeta + L_\zeta{}^* \cdot L_\zeta\right) =$$
$$= \left(\frac{1-t}{t}\right)^2 \Big((Dh(\zeta)|_{\zeta^\perp})^* \cdot \Delta\left(\|\zeta\|^{-2d_i+2}\right) \cdot Dh(\zeta)|_{\zeta^\perp} +$$
$$+ \left\|\Delta\left(\sqrt{d_i}\|\zeta\|^{-d_i}\right) h(\zeta)\right\|^2 \mathrm{Id}_{\zeta^\perp}\Big).$$

The proof follows. □

Lemma 3.3.5. *One has*

$$\left|\det\left[D(\pi_1 \circ \Phi_t)(\hat{h}_t(\zeta),\zeta) \cdot (Id_{\mathcal{H}_{(d)}}, -(D\hat{h}_t(\zeta)|_{\zeta^\perp})^*)\right]\right| =$$
$$= |\det(Id_{\zeta^\perp} + (D\hat{h}_t(\zeta)|_{\zeta^\perp})^* \cdot D\hat{h}_t(\zeta)|_{\zeta^\perp})|\, t^{2n}.$$

Proof. First we find an expression for the term inside the determinant. For short, let

$$\psi = D(\pi_1 \circ \Phi_t)(\hat{h}_t(\zeta),\zeta) \cdot (\mathrm{Id}_{\mathcal{H}_{(d)}}, -(D\hat{h}_t(\zeta)|_{\zeta^\perp})^*).$$

One gets,

$$\left[\frac{\partial}{\partial f}(\pi_1 \circ \Phi_t)(f,\zeta)\right](\dot{f}) = \dot{f} - (1-t)\Delta\left(\frac{\langle\cdot,\zeta\rangle^{d_i}}{\langle\zeta,\zeta\rangle^{d_i}}\right)\dot{f}(\zeta), \tag{3.3.10}$$

and

$$\left[\frac{\partial}{\partial \zeta}(\pi_1 \circ \Phi_t)(f,\zeta)\right](\dot{\zeta}) = \tag{3.3.11}$$
$$-(1-t)\left[\Delta\left(\frac{\langle\cdot,\zeta\rangle^{d_i}}{\langle\zeta,\zeta\rangle^{d_i}}\right) \cdot Df(\zeta)\dot{\zeta} + \Delta\left(\frac{d_i\langle\cdot,\zeta\rangle^{d_i-1}\langle\cdot,\dot{\zeta}\rangle}{\langle\zeta,\zeta\rangle^{d_i}}\right) f(\zeta)\right].$$

Since $\hat{h}_t(\zeta)(\zeta) = h(\zeta)/t$, and $D[\hat{h}_t(\zeta)](\zeta)|_{\zeta^\perp} = Dh(\zeta)|_{\zeta^\perp}$, from (3.3.10) and

(3.3.11) we get

$$\begin{aligned}\psi(\dot f) = \dot f - (1-t)\Delta\left(\frac{\langle\cdot,\zeta\rangle^{d_i}}{\langle\zeta,\zeta\rangle^{d_i}}\right)\dot f(\zeta)+ \\ + (1-t)\Big[\Delta\left(\frac{\langle\cdot,\zeta\rangle^{d_i}}{\langle\zeta,\zeta\rangle^{d_i}}\right)\cdot Dh(\zeta)|_{\zeta^\perp}\cdot (D\hat h_t(\zeta)|_{\zeta^\perp})^*\dot f+ \\ + \Delta\left(\frac{d_i\langle\cdot,\zeta\rangle^{d_i-1}\langle\cdot,(D\hat h_t(\zeta)|_{\zeta^\perp})^*\dot f\rangle}{\langle\zeta,\zeta\rangle^{d_i}}\right)\frac{h(\zeta)}{t}\Big],\end{aligned}$$

for all $\dot f \in \mathcal{H}_{(d)}$. That is, with the notation K_ζ and L_ζ given in (3.3.6) and (3.3.7), we get

$$\begin{aligned}\psi(\dot f) = \dot f - (1-t)\left[\Delta\left(\frac{\langle\cdot,\zeta\rangle^{d_i}}{\langle\zeta,\zeta\rangle^{d_i}}\right)\dot f(\zeta) - \left(\frac{1-t}{t}\right)K_\zeta\big({K_\zeta}^* + {L_\zeta}^*\big)\dot f\right] + \\ + \left(\frac{1-t}{t}\right)^2 L_\zeta\big({K_\zeta}^*+{L_\zeta}^*\big)\dot f\end{aligned} \tag{3.3.12}$$

for all $\dot f \in \mathcal{H}_{(d)}$.

Note that $\psi = \mathrm{Id}_{\mathcal{H}_{(d)}} - \mathcal{L}$, for a certain operator $\mathcal{L}$. Therefore $\det(\psi) = \det((\mathrm{Id}_{\mathcal{H}_{(d)}} - \mathcal{L})|_{\mathrm{Im}\mathcal{L}})$, where last determinant must be understood as the determinant of the linear operator $(\mathrm{Id}_{\mathcal{H}_{(d)}} - \mathcal{L})|_{\mathrm{Im}\mathcal{L}} : \mathrm{Im}\mathcal{L} \to \mathrm{Im}\mathcal{L}$.

The image of $\mathcal{L}$ is decomposed into two orthogonal subspaces, namely:

$$\begin{aligned}C_\zeta &:= \left\{\Delta\left(\frac{\langle\cdot,\zeta\rangle^{d_i}}{\langle\zeta,\zeta\rangle^{d_i}}\right)a :\ a = (a_1,\ldots,a_n)^T \in \mathbb{C}^n\right\};\\ R_\zeta &:= \left\{L_\zeta(w) :\ w \in T_\zeta\mathbb{P}(\mathbb{C}^{n+1})\right\}.\end{aligned}$$

Note that $\mathrm{Im}K_\zeta = C_\zeta \subset \ker {L_\zeta}^*$ and $\mathrm{Im}L_\zeta = R_\zeta \subset \ker {K_\zeta}^*$.

Consider the linear map

$$\tau : \mathbb{C}^n \to C_\zeta, \quad \tau(b) = \Delta\left(\frac{\langle\cdot,\zeta\rangle^{d_i}}{\langle\zeta,\zeta\rangle^{d_i}}\right)\cdot\Delta(\|\zeta\|^{d_i})b, \quad b\in\mathbb{C}^n.$$

Note that, $\tau^{-1}\left(\Delta\left(\frac{\langle\cdot,\zeta\rangle^{d_i}}{\langle\zeta,\zeta\rangle^{d_i}}\right)a\right) = \Delta(\|\zeta\|^{-d_i})\cdot a$. Since

$$\|\Delta\left(\frac{\langle\cdot,\zeta\rangle^{d_i}}{\langle\zeta,\zeta\rangle^{d_i}}\right)a\| = \|\Delta(\|\zeta\|^{-d_i})\cdot a\|,$$

we conclude that τ is a linear isometry between $\mathbb{C}^n$ and C_ζ.

Let

$$\eta : T_\zeta\mathbb{P}(\mathbb{C}^{n+1}) \to R_\zeta, \quad \eta(\cdot) = \frac{\|\zeta\|}{\left\|\Delta\left(\sqrt{d_i}\|\zeta\|^{-d_i}\right)h(\zeta)\right\|} L_\zeta(\cdot).$$

Since

$$\|L_\zeta(w)\| = \left\|\Delta\left(\sqrt{d_i}\|\zeta\|^{-d_i}\right)h(\zeta)\right\| \cdot \frac{\|w\|}{\|\zeta\|},$$

for all $w \in T_\zeta\mathbb{P}(\mathbb{C}^{n+1})$, we get that η is a linear isometry between $T_\zeta\mathbb{P}(\mathbb{C}^{n+1})$ and R_ζ.

Let $\Pi_{C_\zeta}\psi$ and $\Pi_{R_\zeta}\psi$ be the orthogonal projections on C_ζ and R_ζ respectively.

Then $|\det(\psi)|$ is equal to the absolute value of the determinant of

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where $A = \tau^{-1} \circ \Pi_{C_\zeta}\psi|_{C_\zeta} \circ \tau$, $B = \tau^{-1} \circ \Pi_{C_\zeta}\psi|_{R_\zeta} \circ \eta$, $C = \eta^{-1} \circ \Pi_{R_\zeta}\psi|_{C_\zeta} \circ \tau$ and $D = \eta^{-1} \circ \Pi_{R_\zeta}\psi|_{R_\zeta} \circ \eta$.

Straightforward computations show that

$$\begin{aligned}
A &= t\,\mathrm{Id}_{\mathbb{C}^n} + \frac{(1-t)^2}{t}\Delta(\|\zeta\|^{-d_i+1}) \cdot Dh(\zeta)|_{\zeta^\perp} \cdot (Dh(\zeta)|_{\zeta^\perp})^* \cdot \Delta(\|\zeta\|^{-d_i+1});\\
B &= \frac{(1-t)^2}{t}\|\Delta(\sqrt{d_i}\|\zeta\|^{-d_i})h(\zeta)\|\,\Delta(\|\zeta\|^{-d_i+1}) \cdot Dh(\zeta)|_{\zeta^\perp};\\
C &= \left(\frac{1-t}{t}\right)^2 \|\Delta(\sqrt{d_i}\|\zeta\|^{-d_i})h(\zeta)\|\,(Dh(\zeta)|_{\zeta^\perp})^* \cdot \Delta(\|\zeta\|^{-d_i+1});\\
D &= \left(1 + \left(\frac{1-t}{t}\right)^2 \|\Delta(\sqrt{d_i}\|\zeta\|^{-d_i})h(\zeta)\|^2\right)\mathrm{Id}_{\zeta^\perp}.
\end{aligned}$$

Since D is invertible, we may write

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A - BD^{-1}C & B \\ 0 & D \end{pmatrix} \cdot \begin{pmatrix} I & 0 \\ D^{-1}C & I \end{pmatrix},$$

hence $\det\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det D \cdot \det(A - BD^{-1}C)$.

Thus,

$$|\det(\psi)| = t^{2n}\left(1+\left(\frac{1-t}{t}\right)^2\cdot\left\|\Delta\left(\sqrt{d_i}\|\zeta\|^{-d_i}\right)h(\zeta)\right\|^2\right)^{2n}\cdot$$
$$\left|\det\left(\mathrm{Id}_{\mathbb{C}^n}+\frac{\left(\frac{1-t}{t}\right)^2\Delta(\|\zeta\|^{-d_i+1})\cdot Dh(\zeta)|_{\zeta^\perp}\cdot(Dh(\zeta)|_{\zeta^\perp})^*\cdot\Delta(\|\zeta\|^{-d_i+1})}{1+\left(\frac{1-t}{t}\right)^2\cdot\left\|\Delta\left(\sqrt{d_i}\|\zeta\|^{-d_i}\right)h(\zeta)\right\|^2}\right)\right|^2$$

Observe that

$$(Dh(\zeta)|_{\zeta^\perp})^*\cdot\Delta\left(\|\zeta\|^{-d_i+1}\right)^2\cdot Dh(\zeta)|_{\zeta^\perp} = \left(\Delta\left(\|\zeta\|^{-d_i+1}\right)\cdot Dh(\zeta)|_{\zeta^\perp}\right)^*\cdot\left(\Delta\left(\|\zeta\|^{-d_i+1}\right)\cdot Dh(\zeta)|_{\zeta^\perp}\right)$$

Then, proof follows from *Lemma* 3.3.4 and Sylvester theorem (3.3.5). □

Proof of Proposition 3.3.1. The jacobian of $\hat{H}_t:\mathbb{P}(\mathbb{C}^{n+1})\to\mathcal{H}_{(d)}\times\mathbb{P}(\mathbb{C}^{n+1})$ at ζ is given by

$$\left|\det(Id_{\zeta^\perp}+(D\hat{h}_t(\zeta)|_{\zeta^\perp})^*\cdot D\hat{h}_t(\zeta)|_{\zeta^\perp}\right|^{1/2}.$$

Then, the proof follows from *Lemma 3.3.2* and *Lemma 3.3.5.* □

3.4 Numerical Experiments

In this section we present some numerical experiments for $n = 1$ and $d = 7$ that were performed by Carlos Beltrán on the Altamira supercomputer at the Universidad de Cantabria.

Recall from *Theorem 5* that

$$\Theta(h,\eta) = \int_{\zeta\in B(h,\eta)}\frac{\left(\|h\|^2-\|\Delta(\|\zeta\|^{-d_i})h(\zeta)\|^2\right)^{1/2}}{\|\Delta(\|\zeta\|^{-d_i})h(\zeta)\|^{2n-1}}\cdot \cdot\Gamma(\|\Delta(\|\zeta\|^{-d_i})h(\zeta)\|^2/2,n)e^{\|\Delta(\|\zeta\|^{-d_i})h(\zeta)\|^2/2}\,d\zeta.$$

Let

$$\overline{\Theta(h)} = \int_{\zeta\in\mathbb{P}(\mathbb{C}^{n+1})}\frac{1}{\|\Delta(\|\zeta\|^{-d_i})h(\zeta)\|^{2n-1}}\cdot e^{\|\Delta(\|\zeta\|^{-d_i})h(\zeta)\|^2/2}\,d\zeta.$$

(Recall item (d) after the statement of the main theorem).

Note that

$$\sum_{\eta:\, h(\eta)=0} \Theta(h,\eta) \leq \|h\|\,\Gamma(n)\,\overline{\Theta(h)}.$$

Table 3.1 concerns a degree 7 polynomial h, chosen at random with the Bombieri-Weyl distribution. The condition numbers $\mu(h,\eta)$, $\Theta(h,\eta)$ and $\mathrm{vol}(B(h,\eta))$, at each root η of h are computed. Moreover, $\overline{\Theta(h)}$ is computed.

The data of the chosen random polynomial is given by:

$$\begin{aligned}
a_7 &= -0.152840 - i0.757630\\
a_6 &= 1.283080 + i0.357670\\
a_5 &= 2.000560 + i3.302700\\
a_4 &= 13.004500 + i0.203300\\
a_3 &= -1.138140 + i7.094290\\
a_2 &= 3.110090 + i2.618830\\
a_1 &= 0.282940 + -i0.276260\\
a_0 &= -0.316220 + i0.036590,
\end{aligned}$$

One gets $\|h\| = 2.9631$ and $\overline{\Theta(h)} = 7.624646$.

Roots in $\mathbb{C}$	$\mu(h,\cdot)$	$\Theta(h,\cdot)$	$\mathrm{vol}(B(h,\cdot))$
$3.260883 - i1.658800$	1.712852	1.487095	0.140509π
$-2.357860 - i1.329208$	1.738380	1.728768	0.138576π
$-0.210068 + i1.868947$	1.608231	1.586398	0.144054π
$0.227994 - i0.782004$	1.909433	1.544021	0.125685π
$-0.044701 + i0.384342$	3.231554	3.152883	0.147277π
$-0.308283 + i0.049618$	3.183603	2.793696	0.152433π
$0.213950 - i0.068700$	2.948318	2.647258	0.151466π

Table 3.1: Degree 7 random polynomial.

In *Figure 3.1* we have plotted, using GNU Octave, the basins $B(h,\eta)$ at each root η of the chosen random polynomial h are plotted, in $\mathbb{C}$ and in the Riemann sphere,.

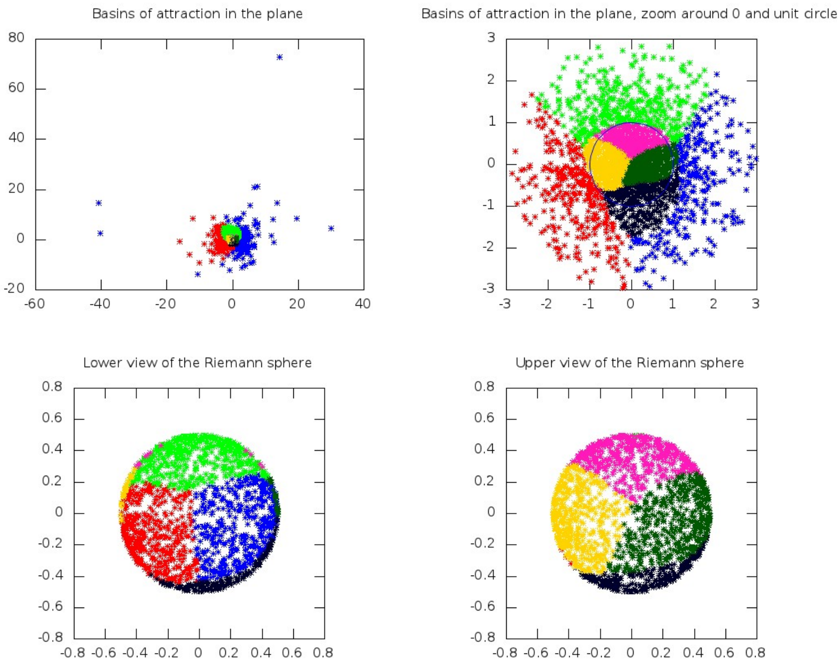


Figure 3.1: The basins $B(h, \eta)$ in $\mathbb{C}$ and in the Riemann sphere of the degree 7 random polynomial (GNU Octave).

In *Table 3.2* the same quantities are computed for the polynomial given by $a_0 = -1$, $a_1 = \ldots = a_6 = 0$, $a_7 = 1$. In this case the roots are the 7th roots of unity, and it is not difficult to see that the actual values of $\mu(h, \eta)$, $\Theta(h, \eta)$ and $\mathrm{vol}(B(h, \eta))$ are constant at the roots of h by symmetry. This example illustrate the extent of accuracy of the computations.

Roots in $\mathbb{C}$	$\mu(h, \cdot)$	$\Theta(h, \cdot)$	$\mathrm{vol}(B(h, \cdot))$
$-0.900969 + i0.433884$	3.023716	2.210393	0.128982π
$-0.900969 - i0.433884$	3.023716	2.624508	0.153846π
$-0.222521 + i0.974928$	3.023716	2.326541	0.135198π
$-0.222521 - i0.974928$	3.023716	2.371825	0.141414π
$1.000000 + i0.000000$	3.023716	2.867733	0.156954π
$0.623490 + i0.781831$	3.023716	2.136386	0.135198π
$0.623490 - i0.781831$	3.023716	2.551867	0.148407π

Table 3.2: $h(z) = z^7 - 1$.

In this case we get $\|h\| = \sqrt{2}$ and $\overline{\Theta(h)} = 13.157546$.

The errors for the root of unity case in the third column are of the order of 25%. But 25% does not seem enough to explain the variation in the computed quantities in the third column of the random example where the ratio of the max to min is greater than 2. So it is likely that they are not all equal. On the other hand, the ratios of the volumes of the basins in the fourth columns of the random and roots of unity examples do seem of the same order of magnitude. So perhaps they are all equal? Also,the graphics of the basins are very encouraging in the random case. There appear to be 7 connected regions with a root in each. So there is some hope that this is true in general. That is there may generically be a root in each connected component of the basins and all these basins may have equal volume. This would be very interesting and would be very good start on understanding the integrals. It would be good to have some more experiments and even better some theorems.

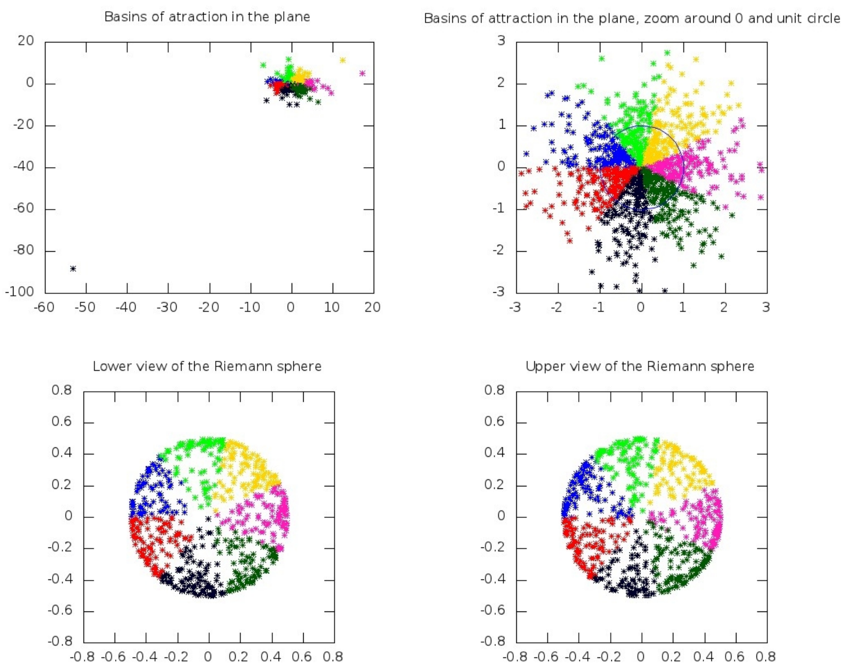


Figure 3.2: The basins $B(h, \eta)$ in $\mathbb{C}$ and in the Riemann sphere for $h(z) = z^7 - 1$ (GNU Octave).

Appendices

Appendix A

Stochastic Perturbations and Smooth Condition Numbers

In this appendix it is defined a new condition number adapted to directionally uniform perturbations in a general framework of maps between Riemannian manifolds. The definitions and theorems can be applied to a large class of problems. The relation with the classical condition number in many interesting examples is studied.

A.1 Introduction and Main Result

Let $\mathcal{X}$ and $\mathcal{Y}$ be two real (or complex) Riemannian manifolds of real dimensions m and n ($m \geq n$) associated respectively to some computational problem, where $\mathcal{X}$ is the space of *inputs* and $\mathcal{Y}$ is the space of *outputs*.

Recall fromt the *Introduction* that $\mathcal{V} \subset \mathcal{X} \times \mathcal{Y}$ is the solution variety; $\pi_1 : \mathcal{V} \to \mathcal{X}$ and $\pi_2 : \mathcal{V} \to \mathcal{Y}$ are the canonical projections; Σ' and Σ are the ill-posed variety and the discriminant variety respectively.

When $\dim \mathcal{V} = \dim \mathcal{X}$, for each $(x, y) \in \mathcal{V} \setminus \Sigma'$, we have the solution map $\mathscr{S}(x, y) : U_x \to U_y$ defined between some neighborhoods U_x and U_y of $x \in \mathcal{X}$ and $y \in \mathcal{Y}$ respectively.

Let us denote by $\langle \cdot, \cdot \rangle_x$ and $\langle \cdot, \cdot \rangle_y$ the Riemannian (or Hermitian) inner product in the tangent spaces $T_x\mathcal{X}$ and $T_y\mathcal{Y}$ at x and y respectively.

A. STOCHASTIC PERTURBATIONS AND SMOOTH CONDITION NUMBERS

Recall from the *Introduction* that the condition number at $(x, y) \in \mathcal{V} \setminus \Sigma'$ is given by:

$$\mu(x, y) := \max_{\substack{\dot{x} \in T_x \mathcal{X} \\ \|\dot{x}\|_x^2 = 1}} \|D\mathscr{S}(x)\dot{x}\|_y. \tag{A.1.1}$$

See the *Introduction* for references about the role of the condition number in numerical analysis and complexity of algorithms.

In many practical situations, there exists a discrepancy between worst case theoretical analysis and observed accuracy of an algorithm. There exist several approaches that attempt to rectify this discrepancy. Among them we find *average-case analysis* (see Edelman [1989], Smale [1985]) and *smooth analysis* (see Spielman & Teng [2002], Bürgisser *et al.* [2006], Wschebor [2004]). For a comprehensive review on this subject with historical notes see Bürgisser [2009].

In many problems, the space of inputs has a much larger dimension than the one of the space of outputs ($m \gg n$). Then, it is natural to assume that infinitesimal perturbations of the input will produce drastic changes in the output only when they are performed in a few directions. Then, a possibly different approach to analyze accuracy of algorithms is to replace "worst direction" by a certain mean over all possible directions. This alternative was already suggested and studied in Weiss *et al.* [1986] in the case of linear system solving $Ax = b$, and more generally, in Stewart [1990] in the case of matrix perturbation theory, where the first-order perturbation expansion is assumed to be random.

In this chapter we extend this approach to a large class of computational problems, restricting ourselves to the case of directionally uniform perturbations.

Generalizing the concept introduced in Weiss *et al.* [1986] and Stewart [1990], we define the *pth-stochastic condition number* at (x, y) as:

$${\mu_{st}}^{[p]}(x, y) := \left[\frac{1}{\mathrm{vol}(S_x^{m-1})} \int_{\dot{x} \in S_x^{m-1}} \|D\mathscr{S}(x)\dot{x}\|_y^p \, dS_x^{m-1}(\dot{x})\right]^{1/p}, \quad (p = 1, 2, \ldots), \tag{A.1.2}$$

where $\mathrm{vol}(S_x^{m-1}) = \frac{2\pi^{m/2}}{\Gamma(m/2)}$ is the measure of the unit sphere S_x^{m-1} in $T_x\mathcal{X}$, and dS_x^{m-1} is the induced volume element. We will be mostly interested in the case $p = 2$, which we simply write μ_{st} and call it *stochastic condition number*.

Before stating the main theorem, we define the *Frobenius condition number* as:

$$\mu_F(x,y) := \|D\mathscr{S}(x)\|_F = \sqrt{\sigma_1^2 + \cdots + \sigma_n^2}$$

where $\|\cdot\|_F$ is the Frobenius norm and $\sigma_1, \ldots, \sigma_n$ are the singular values of the condition operator. Note that $\mu_F(x,y)$ is a smooth function in $\mathcal{V} \setminus \Sigma'$, where its differentiability class depends on the differentiability class of G.

Theorem 6.

$$\mu_{st}{}^{[p]}(x,y) = \frac{1}{\sqrt{2}} \left[\frac{\Gamma\left(\frac{m}{2}\right)}{\Gamma\left(\frac{m+p}{2}\right)} \right]^{1/p} \cdot \mathbb{E}(\|\eta_{\sigma_1,\ldots,\sigma_n}\|^p)^{1/p},$$

where $\|\cdot\|$ is the Euclidean norm in $\mathbb{R}^n$ and $\eta_{\sigma_1,\ldots,\sigma_n}$ is a centered Gaussian vector in $\mathbb{R}^n$ with diagonal covariance matrix Diag$(\sigma_1^2, \ldots, \sigma_n^2)$.
In particular, for $p = 2$

$$\mu_{st}(x,y) = \frac{\mu_F(x,y)}{\sqrt{m}}. \tag{A.1.3}$$

Remark A.1.1. Since $\mu(x,y) \leq \mu_F(x,y) \leq \sqrt{n} \cdot \mu(x,y)$, we have from (A.1.3) that

$$\frac{1}{\sqrt{m}} \cdot \mu(x,y) \leq \mu_{st}(x,y) \leq \sqrt{\frac{n}{m}} \cdot \mu(x,y).$$

This result is most interesting when $m \gg n$, for in that case $\mu_{st}(x,y) \ll \mu(x,y)$. Thus, in these cases one may expect much better stability properties than those predicted by classical condition numbers.

Remark A.1.2. In many situations, one needs to analyze how the condition number varies in order to study (or to improve) the accuracy of an algorithm. In this way, the replacement of the usual non-smooth condition number μ given in (A.1.1) by a smooth one, has an important theoretical and practical application.

In numerical analysis, many authors are interested in relative errors. Thus, when $(\mathcal{X}, \langle\cdot,\cdot\rangle_{\mathcal{X}})$ and $(\mathcal{Y}, \langle\cdot,\cdot\rangle_{\mathcal{Y}})$ are real (or complex) finite dimensional vector spaces with an inner (or Hermitian) product, instead of considering the (absolute) condition number (A.1.1), one can take the *relative condition number* defined as:

$$\mu_{rel}(x,y) := \frac{\|x\|_{\mathcal{X}}}{\|y\|_{\mathcal{Y}}} \cdot \mu(x,y), \qquad x \neq 0,\ y \neq 0;$$

and the *relative Frobenius condition number* as:

$$\mu_{rel\,F}(x,y) := \frac{\|x\|_{\mathcal{X}}}{\|y\|_{\mathcal{Y}}} \cdot \mu_F(x,y), \qquad x \neq 0,\, y \neq 0,$$

where $\|\cdot\|_{\mathcal{X}}$ and $\|\cdot\|_{\mathcal{Y}}$ are the respective induced norms. In the same way, we define the *relative pth-stochastic condition number* as

$$\mu_{rel\,st}^{[p]}(x,y) := \frac{\|x\|_{\mathcal{X}}}{\|y\|_{\mathcal{Y}}} \cdot \mu_{st}{}^{[p]}(x,y), \quad (p = 1, 2, \ldots). \tag{A.1.4}$$

For the case $p = 2$ we simply write $\mu_{rel\,st}$ and call it *relative stochastic condition number.*

In this case, we can define Riemannian structures on $\mathcal{X} \setminus \{0\}$ and $\mathcal{Y} \setminus \{0\}$ in the following way: for each $x \in \mathcal{X}$, $x \neq 0$, and $y \in \mathcal{Y}$, $y \neq 0$, we define

$$\langle \cdot, \cdot \rangle_x := \frac{\langle \cdot, \cdot \rangle_{\mathcal{X}}}{\|x\|_{\mathcal{X}}^2}, \quad \text{and} \quad \langle \cdot, \cdot \rangle_y := \frac{\langle \cdot, \cdot \rangle_{\mathcal{Y}}}{\|y\|_{\mathcal{Y}}^2}.$$

Notice that, in these Riemannian structures the usual condition number defined in (A.1.1) turns to be the relative condition number defined before. Then, *Theorem 6* remains true if one exchanges the (absolute) condition number by the relative condition number. In particular,

$$\mu_{rel\,st}(x,y) := \frac{\mu_{rel\,F}(x,y)}{\sqrt{m}}.$$

A.2 Componentwise Analysis

In the case $\mathcal{Y} = \mathbb{R}^n$ we define the *kth-componentwise condition number* at $(x,y) \in \mathcal{V} \setminus \Sigma'$ as:

$$\mu(x,y;k) := \max_{\substack{\dot{x} \in T_x\mathcal{X} \\ \|\dot{x}\|_x^2 = 1}} |(D\mathscr{S}(x)\dot{x})_k|, \qquad (k = 1, \ldots, n), \tag{A.2.1}$$

where $|\cdot|$ is the absolute value and w_k indicates the kth-component of the vector $w \in \mathbb{R}^n$.

Following Weiss *et al.* [1986] for the linear case, we define the *pth-stochastic kth-componentwise condition number* as:

$$\mu_{st}{}^{[p]}(x, y; k) := \left[\frac{1}{\mathrm{vol}(S_x^{m-1})} \int_{\dot{x} \in S_x^{m-1}} |(D\mathscr{S}(x)\dot{x})_k|^p \; dS_x^{m-1}(\dot{x})\right]^{1/p}, \quad (p = 1, 2, \ldots). \tag{A.2.2}$$

Then we have:

Proposition A.2.1.

$$\mu_{st}{}^{[p]}(x, y; k) = \left[\frac{1}{\sqrt{\pi}} \cdot \frac{\Gamma\left(\frac{m}{2}\right)}{\Gamma\left(\frac{m+p}{2}\right)} \cdot \Gamma\left(\frac{p+1}{2}\right)\right]^{1/p} \cdot \mu(x, y; k).$$

In particular,

$$\mu_{st}(x, y; k) = \frac{\mu(x, y; k)}{\sqrt{m}}.$$

Proof. Observe that $\mu_{st}{}^{[p]}(x, y; k)$ is the pth-stochastic condition number for the problem of finding the kth-component of $G = (G_1, \ldots, G_n) : \mathfrak{X} \to \mathbb{R}^n$. *Theorem 6* applied to G_k yields

$$\mu_{st}{}^{[p]}(x, y; k) = \frac{1}{\sqrt{2}} \left[\frac{\Gamma\left(\frac{m}{2}\right)}{\Gamma\left(\frac{m+p}{2}\right)}\right]^{\frac{1}{p}} \cdot \mathbb{E}(|\eta_{\sigma_1}|^p)^{1/p}$$

where $\sigma_1 = \|D\mathscr{S}_k(x)\| = \mu(x, y; k)$. Then,

$$\mathbb{E}(|\eta_{\sigma_1}|^p)^{1/p} = \mu(x, y; k) \cdot \mathbb{E}(|\eta_1|^p)^{1/p},$$

where η_1 is a standard normal in $\mathbb{R}$. Finally,

$$\mathbb{E}(|\eta_1|^p) = \frac{2}{\sqrt{2\pi}} \int_0^\infty \rho^p e^{-\rho^2/2} \, d\rho = \frac{2}{\sqrt{2\pi}} 2^{\frac{p-1}{2}} \Gamma(\frac{p+1}{2}),$$

and the proposition follows. $\square$

A.3 Proof of the main Theorem

In the case of complex manifolds, the condition matrix turns to be an $n \times n$ complex matrix. In what follows, we identify it with the associated $2n \times 2n$ real matrix. We focus on the real case.

The main theorem follows immediately from *Lemma A.3.1* and *Proposition A.3.1* below.

Lemma A.3.1. *Let η be a Gaussian standard random vector in $\mathbb{R}^m$. Then*

$$\mu_{st}{}^{[p]}(x,y) = \frac{1}{\sqrt{2}} \left[\frac{\Gamma\left(\frac{m}{2}\right)}{\Gamma\left(\frac{m+p}{2}\right)} \right]^{1/p} \cdot \left[\mathbb{E}(\|D\mathscr{S}(x)\eta\|^p)\right]^{1/p},$$

where $\mathbb{E}$ is the expectation operator and $\|\cdot\|$ is the Euclidean norm in $\mathbb{R}^n$.

Proof. Let $f : \mathbb{R}^m \to \mathbb{R}$ be the continuous function given by

$$f(v) = \|D\mathscr{S}(x)v\|.$$

Then,

$$\left[\mathbb{E}(\|D\mathscr{S}(x)\eta\|^p)\right]^{1/p} = \left[\frac{1}{(2\pi)^{m/2}} \int_{\mathbb{R}^m} f(v)^p \cdot e^{-\|v\|^2/2}\, dv \right]^{1/p}.$$

Integrating in polar coordinates, we get

$$\mathbb{E}(\|D\mathscr{S}(x)\eta\|^p) = \frac{I_{m+p-1}}{(2\pi)^{m/2}} \cdot \int_{S^{m-1}} f^p \, dS^{m-1}, \tag{A.3.1}$$

where

$$I_j = \int_0^{+\infty} \rho^j \, e^{-\rho^2/2}\, d\rho, \quad j \in \mathbb{N}.$$

Making the change of variable $u = \rho^2/2$ we obtain

$$I_j = 2^{\frac{j-1}{2}} \Gamma(\frac{j+1}{2}),$$

therefore

$$I_{m+p-1} = 2^{\frac{m+p-2}{2}} \cdot \Gamma\left(\frac{m+p}{2}\right). \tag{A.3.2}$$

Then, joining together (A.3.1) and (A.3.2) we obtain the result. $\square$

Proposition A.3.1. *If η is a Gaussian standard random vector in $\mathbb{R}^m$, then*

$$\mathbb{E}(\|D\mathscr{S}(x)\eta\|^p) = \mathbb{E}(\|\eta_{\sigma_1,\ldots,\sigma_n}\|^p),$$

where $\eta_{\sigma_1,\ldots,\sigma_n}$ is a centered Gaussian vector in $\mathbb{R}^n$ with diagonal covariance matrix $Diag(\sigma_1^2,\ldots,\sigma_n^2)$, and $\sigma_1,\ldots,\sigma_n$ are the singular values of $D\mathscr{S}(x)$.

Proof. Let $D\mathscr{S}(x) = UDV$ be a singular value decomposition of $D\mathscr{S}(x)$, where V and U are orthogonal transformations of $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively, and $D := Diag(\sigma_1,\ldots,\sigma_n)$. By the invariance of the Gaussian distribution under the action of the orthogonal group in $\mathbb{R}^m$, $V\eta$ is again a Gaussian standard random vector in $\mathbb{R}^m$. Then,

$$\mathbb{E}(\|D\mathscr{S}(x)\eta\|^p) = \mathbb{E}(\|UD\eta\|^p),$$

and by the invariance under the action of the orthogonal group of the Euclidean norm, we get

$$\mathbb{E}(\|D\mathscr{S}(x)\eta\|^p) = \mathbb{E}(\|D\eta\|^p).$$

Finally $D\eta$ is a centered Gaussian vector in $\mathbb{R}^n$ with covariance matrix $Diag(\sigma_1^2,\ldots,\sigma_n^2)$, and the proposition follows. For the case $p = 2$,

$$\mu_{st}(x,y) = \left[\mathbb{E}\left(\sigma_1^2\eta_1^2 + \ldots + \sigma_n^2\eta_n^2\right)\right]^{1/2},$$

where $\eta_1,\ldots,\eta_n$ are i.i.d. standard normal in $\mathbb{R}$. Then,

$$\mu_{st}(x,y) = \left(\sum_{i=1}^{n}\sigma_i^2\right)^{1/2} = \mu_F(x,y).$$

$\square$

A.4 Examples

In this section we will compute the stochastic condition number for different problems: systems of linear equations, eigenvalue and eigenvector problems, finding kernels of linear transformations and solving polynomial systems of equations.

The first two have been computed in Stewart [1990] and are an easy consequence of *Theorem 6* and the usual condition number μ.

The computations of μ for the case of systems of linear equations, eigenvalue and eigenvector problems, and solving polynomial systems of equations are fairly well-known. However, as far as we know, previous results of μ for the problem of finding kernels of linear transformations only offers bounds (see Kahan [2000], Stewart & Sun [1990], Beltrán & Pardo [2007]). In *Section A.4.3* we gave an explicit computation of μ for this problem.

In what follows, we will drop the output in the notation of condition number when the input-output map is univalued.

A.4.1 Systems of Linear Equations

We consider the problem of solving for $y \in \mathbb{R}^n$ the system of linear equations $Ay = b$, $y \neq 0$, where $A \in \mathbb{R}^{n\times n}$ (the space of $n \times n$ real matrices), and $b \in \mathbb{R}^n$.

If we assume that b is fixed, then, we can consider the input space $\mathcal{X} = \mathbb{R}^{n\times n}$ equipped with the Frobenius inner product

$$\langle A, B\rangle_F = \mathrm{trace}(AB^t), \tag{A.4.1}$$

where B^t is the transpose of B, and the output space $\mathcal{Y} = \mathbb{R}^n$ equipped with the Euclidean inner product. It is easy to see that Σ is the subset of non-invertible matrices. Then, the map $G : \mathbb{R}^{n\times n}\backslash\Sigma \to \mathbb{R}^n$ is globally defined and differentiable, namely

$$G(A) = A^{-1}b \; (= y).$$

By implicit differentiation,

$$D\mathscr{S}(A)\dot{A} = -A^{-1}\dot{A}y. \tag{A.4.2}$$

Is easy to see from (A.4.2) that

$$\mu(A) = \|A^{-1}\| \cdot \|y\|.$$

Let H be the orthogonal complement of $\ker D\mathscr{S}(A)$, i.e. H is the set of rank one matrices of the form uy^t, $u \in \mathbb{R}^n$, where y^t denotes the transpose of $y \in \mathbb{R}^n$. Then, the map $u \mapsto uy^t/\|y\|$ is a linear isometry between $\mathbb{R}^n$ and H. Under this identification, is easy to see from (A.4.2) that $D\mathscr{S}(A)|_H$ coincides with the map $-\|y\| \cdot A^{-1}$, from where we conclude,

$$\mu_F(A) = \|A^{-1}\|_F \cdot \|y\|.$$

Then, from *Theorem 6* we get

$$\mu_{st}(A) = \frac{\|A^{-1}\|_F \cdot \|y\|}{n},$$

and therefore

$$\mu_{st}(A) \leq \frac{\mu(A)}{\sqrt{n}}. \tag{A.4.3}$$

A similar result was proved in Stewart [1990].

For the general case, we consider $\mathcal{X} = \mathbb{R}^{n\times n} \times \mathbb{R}^n$ equipped with the product metric structure of the Frobenius inner product in $\mathbb{R}^{n\times n}$ and the Euclidean inner product in $\mathbb{R}^n$. Then,
$G : \mathbb{R}^{n\times n} \setminus \Sigma \times \mathbb{R}^n \to \mathbb{R}^n$ satisfies $G(A, b) = A^{-1}b$.
Similar to the preceding case, we have $\mu(A, b) = \|A^{-1}\| \cdot \sqrt{1 + \|y\|^2}$ and $\mu_F(A, b) = \|A^{-1}\|_F \cdot \sqrt{1 + \|y\|^2}$. Again from *Theorem 6* we get

$$\mu_{st}(A, b) = \frac{\|A^{-1}\|_F \cdot \sqrt{1 + \|y\|^2}}{\sqrt{n^2 + n}},$$

and therefore

$$\mu_{st}(A, b) \leq \frac{\mu(A, b)}{\sqrt{n + 1}}.$$

For the kth-componentwise condition number, we have that

$$\mu_{st}{}^{[p]}((A, b); k) = \left[\frac{1}{\sqrt{\pi}} \cdot \frac{\Gamma\left(\frac{n^2+n}{2}\right)}{\Gamma\left(\frac{n^2+n+p}{2}\right)} \cdot \Gamma\left(\frac{p+1}{2}\right)\right]^{1/p} \cdot \mu((A, b); k),$$

and
$$\mu_{st}((A,b);k) = \frac{\mu((A,b);k)}{\sqrt{n^2+n}}.$$

A similar result was proved in Weiss *et al.* [1986], where the average in (A.2.2) is performed over the unit ball instead of the unit sphere.

In Edelman [1989], it is proved that the expected value of the relative condition number $\mu_{rel}(A) = \|A\| \cdot \|A^{-1}\|$ of a random matrix A whose elements are i.i.d standard normal, satisfies:

$$\mathbb{E}(\log \mu_{rel}(A)) = \log n + c + o(1),$$

as $n \to \infty$, where $c \approx 1.537$. If we consider the relative stochastic condition number defined in (A.1.4), we get from (A.4.3)

$$\mathbb{E}(\log \mu_{rel\,st}(A)) \leq \frac{1}{2}\log n + c + o(1),$$

as $n \to \infty$.

A.4.2 Eigenvalue and Eigenvector Problem

In this subsection we follow the approach given in Shub & Smale [1996]. However, we alert the reader that in *Chapter 1* we developed a new approach for the eigenvalue problem which exploit other natural symmetries of the problem.

We focus on the complex case. The real case is analogue.

We consider the problem of solving for $(\lambda, v) \in \mathbb{C}\times\mathbb{C}^n$ the system of equations $(\lambda I_n - A)v = 0$, $v \neq 0$, where $A \in \mathbb{C}^{n\times n}$ (the space of $n \times n$ complex matrices).[1]

Since this system of equations is homogenous in v, we define the solution variety associated to this problem as:

$$\mathcal{V} = \{(A, v, \lambda) \in \mathbb{C}^{n\times n} \times \mathbb{P}(\mathbb{C}^n) \times \mathbb{C} :\ (\lambda I_n - A)v = 0\},$$

where $\mathbb{P}(\mathbb{C}^n)$ denotes the projective space associated with $\mathbb{C}^n$.

[1] In *Chapter 1* we define a different framework for the eigenvalue problem.

Let $\mathcal{X} = \mathbb{C}^{n\times n}$ be equipped with the Frobenius Hermitian inner product, i.e. the complex analogue of (A.4.1), and $\mathcal{Y} = \mathbb{P}(\mathbb{C}^n) \times \mathbb{C}$ be equipped with the canonical product metric structure.

Then, for $(A, v, \lambda) \in \mathcal{V} \setminus \Sigma'$, i.e. when λ is a simple eigenvalue (cf. Wilkinson Wilkinson [1972]), the condition linear operators $D\mathscr{S}_1$ and $D\mathscr{S}_2$ associated with the eigenvector and eigenvalue problem are:

$$D\mathscr{S}_1(A)\dot{A} = (\pi_{v^\perp}(\lambda I_n - A)|_{v^\perp})^{-1}\left(\pi_{v^\perp}\dot{A}v\right) \quad \text{and} \quad D\mathscr{S}_2(A)\dot{A} = \frac{\langle \dot{A}v, u\rangle}{\langle v, u\rangle},$$

where $\pi_{v^\perp}$ denotes the orthogonal projection onto $v^\perp$, and u is some left eigenvector associated with λ, that is, $u^*A = \overline{\lambda}u^*$.

The associated condition numbers are:

$$\mu_1(A, v) = \left\|(\pi_{v^\perp}(\lambda I_n - A)|_{v^\perp})^{-1}\right\| \quad \text{and} \quad \mu_2(A, \lambda) = \frac{\|v\| \cdot \|u\|}{|\langle v, u\rangle|}. \tag{A.4.4}$$

From our *Theorem 6*, we get the respective stochastic condition numbers:

$$\mu_{1st}(A, v) = \frac{1}{n}\left\|(\pi_{v^\perp}(\lambda I_n - A)|_{v^\perp})^{-1}\right\|_F \leq \frac{1}{\sqrt{n}}\mu_1(A, v),$$

$$\mu_{2st}(A, \lambda) = \frac{1}{n}\mu_2(A, \lambda).$$

A similar result for $\mu_{2st}(A, \lambda)$ was proved in Stewart Stewart [1990].

A.4.3 Finding Kernels of Linear Transformations

For the sake of completeness of the exposition we focus on the complex case. All ideas carry over naturally on the real case.

Let $\mathbb{C}^{k\times p}$ be the linear space of $k \times p$ complex matrices with the Frobenius Hermitian inner product, and $\mathcal{R}_r \subset \mathbb{C}^{k\times p}$ be the subset of matrices of rank r. Given $A \in \mathcal{R}_r$ we consider the problem of finding the subspace F of $\mathbb{C}^p$ such that $Ax = 0$ for all $x \in F$, i.e. finding the kernel subspace $\ker(A)$ of A. For this purpose, we introduce the *Grassmannian* manifold $\mathbb{G}_{p,\ell}$ of complex subspaces of dimension ℓ in $\mathbb{C}^p$, where $\ell = p - r$ is the dimension of $\ker(A)$.

The input space $\mathcal{X} = \mathcal{R}_r$ is a smooth submanifold of $\mathbb{C}^{k\times p}$ of complex dimension $(k+p)r - r^2$ (see Dedieu [2006]). Thus, it has a natural Hermitian structure induced by the Frobenius Hermitian inner product on $\mathbb{C}^{k\times p}$.

In what follows, we identify $\mathbb{G}_{p,\ell}$ with the quotient $\mathbb{S}_{p,\ell}/\mathcal{U}_\ell$ of the *Stiefel* manifold

$$\mathbb{S}_{p,\ell} := \{M \in \mathscr{M}_{p,\ell}(\mathbb{C}) : \, M^* M = I\}$$

by the unitary group $\mathcal{U}_\ell \subset \mathscr{M}_\ell(\mathbb{C})$, which acts on the right of $\mathbb{S}_{p,\ell}$ in the natural way (see Dedieu [2006]). Then, the complex dimension of the output space $\mathcal{Y} = \mathbb{G}_{p,\ell}$ is $(p-r)r$. (We will use the same letter to represent an element of $\mathbb{S}_{p,\ell}$ and its class in $\mathbb{G}_{p,\ell}$).

The manifold $\mathbb{S}_{p,\ell}$ has a canonical Riemannian structure induced by the real part of the Frobenius Hermitian structure in $\mathscr{M}_{p,\ell}(\mathbb{C})$. On the other hand, $\mathcal{U}_\ell$ is a Lie group of isometries acting on $\mathbb{S}_{p,\ell}$. Therefore, $\mathbb{G}_{p,\ell}$ is a homogeneous space (see Gallot *et al.* [2004]), with a natural Riemannian structure that makes the quotient projection $\pi : \mathbb{S}_{p,\ell} \to \mathbb{G}_{p,\ell}$ a Riemannian submersion. More precisely, the orbit of $M \in \mathbb{S}_{p,\ell}$ under the action of the unitary group $\mathcal{U}_\ell$, namely, $\pi^{-1}(M) = \{MU : \, U \in \mathcal{U}_\ell\}$, defines a smooth submanifold of $\mathbb{S}_{p,\ell}$. In this way, the tangent space $T_M\mathbb{S}_{p,\ell}$ splits into two orthogonally complementary subspaces, namely,

$$T_M\mathbb{S}_{p,\ell} = T_M\pi^{-1}(M) \oplus \left(T_M\pi^{-1}(M)\right)^{\perp},$$

where $T_M\pi^{-1}(M)$ is the tangent space of $\pi^{-1}(M)$ at M. Then, we can naturally identify the tangent space $T_M\mathbb{G}_{p,\ell}$ with $(T_M\pi^{-1}(M))^{\perp}$ with the inherited Riemannian structure induced by $\mathbb{S}_{p,\ell}$. Moreover, in this fashion, we can carry out all computations over the quotient manifold $\mathbb{G}_{p,\ell}$ onto $\mathbb{S}_{p,\ell}$.

To compute the derivative of the input-output map $G : \mathcal{R}_r \to \mathbb{G}_{p,\ell}$ which maps A onto $\ker(A)$, notice that if $M \in \mathbb{S}_{p,\ell}$ is any representative in $\pi^{-1}(\ker(A))$, then $AM = 0$. Then, implicit differentiation in the lift $\mathbb{S}_{p,\ell}$ yields

$$\dot{A}M + A(D\mathscr{S}(A)\dot{A}) = 0,$$

where $\dot{A} \in T_A\mathcal{R}_r$, and $D\mathscr{S}(A)\dot{A} \in T_M\mathbb{G}_{p,\ell}$. Then,

$$D\mathscr{S}(A)\dot{A} = -A^{\dagger}\dot{A}M, \tag{A.4.5}$$

where $A^{\dagger}$ is the Moore-Penrose inverse of A.

We have concluded that the condition operator $D\mathscr{S}(A)$ is a linear map from $T_A\mathcal{R}_r$ (with the Hermitian structure induced by $\mathscr{M}_{k,p}(\mathbb{C})$) onto $(T_M\pi^{-1}(M))^{\perp}$ (with the inherited Riemannian structure of $\mathbb{S}_{p,\ell}$), and given by equation (A.4.5).

One way to compute the singular values of the condition operator described in (A.4.5), is to take an orthonormal basis in $\mathbb{C}^{k\times p}$ which diagonalizes A. From the singular value decomposition, there exists positive numbers $\sigma_1 \geq \cdots \geq \sigma_r > 0$ and orthonormal basis $\{u_1, \ldots, u_k\}$ of $\mathbb{C}^k$ and $\{v_1, \ldots, v_p\}$ of $\mathbb{C}^p$, such that, $A = \sum_{i=1}^{r} \sigma_i u_i v_i^*$ and $A^{\dagger} = \sum_{i=1}^{r} \sigma_i^{-1} v_i u_i^*$. Here w^* denotes the conjugate transpose of the vector w. Thus, $\{u_i v_j^* : i = 1, \ldots, k;\ j = 1, \ldots, p\}$ is an orthonormal basis of $\mathbb{C}^{k\times p}$ which diagonalizes A. In this basis the tangent space $T_A\mathcal{R}_r$ is the orthogonal complement of the subspace generated by $\{u_i v_j^* : i = r+1, \ldots, k;\ j = r+1, \ldots, p\}$.

Acting by an element $U \in \mathcal{U}_\ell$, if necessary, one can assume $M = \sum_{h=1}^{\ell} v_{h+r} e_h^*$, where $\{e_1, \ldots, e_\ell\}$ is the canonical basis of $\mathbb{C}^\ell$. Observe that $\|A^{\dagger}\dot{A}M\|_F \leq \|A^{\dagger}\| \cdot \|\dot{A}M\|_F$. Then,

$$\mu(A) = \|A^{\dagger}\|,$$

where the maximum is attained, for example, at $\dot{A} = u_r v_{r+1}^* \in T_A\mathcal{R}_r$.

Observe that $\mu_F(A)^2 = \sum_{i,j} \|D\mathscr{S}(A) u_i v_j^*\|_F^2$, where the sum runs over all elements $u_i v_j^* \in T_A\mathcal{R}_r$. As $u_i v_j^* \in \ker D\mathscr{S}(A)$, for $i = r+1, \ldots, p$ and $j = 1, \ldots, k$, then,

$$\mu_F(A)^2 = \sum_{i=1}^{r}\sum_{j=1}^{p} \|A^{\dagger} u_i v_j^* M\|_F^2 = \sum_{i=1}^{r}\sum_{j=r+1}^{p} \|\sigma_i^{-1} v_i e_{j-r}^*\|_F^2 = (p-r)\cdot\sum_{i=1}^{r}\sigma_i^{-2}.$$

That is,

$$\mu_F(A) = \sqrt{p-r}\cdot\|A^{\dagger}\|_F.$$

From our *Theorem 6*,

$$\mu_{st}(A) = \frac{\sqrt{p-r}}{\sqrt{(k+p-r)r}} \cdot \|A^{\dagger}\|_F \leq \sqrt{\frac{p(p-r)}{(k+p-r)r}} \cdot \mu(A).$$

In Beltrán [2011], it is proved that

$$\mathbb{E}(\log \mu_{rel}(A) : \, A \in \mathcal{R}_r) \leq \log \left[\frac{k+p-r}{k+p-2r+1}\right] + 2.6,$$

where the expected value is computed with respect to the normalized naturally induced measure in $\mathcal{R}_r$. Our *Theorem 6* immediately yields a bound for the stochastic relative condition number, namely,

$$\mathbb{E}(\log \mu_{rel\,st}(A) : \, A \in \mathcal{R}_r) \leq \frac{1}{2} \log \left[\frac{(k+p-r)r}{(k+p-2r+1)^2 p(p-r)}\right] + 2.6.$$

A.4.4 Finding Roots Problem I: Univariate Polynomials

We start with the case of one polynomial in one complex variable. Let $\mathcal{X} = \mathcal{P}_d = \{f \, : \, f(z) = \sum_{i=0}^{d} f_i z^i, \; f_i \in \mathbb{C}\}$. Identifying $\mathcal{P}_d$ with $\mathbb{C}^{d+1}$, we can define two standard Hermitian inner products in the space $\mathcal{P}_d$:

- Weyl inner product:

$$\langle f, g \rangle_W := \sum_{i=0}^{d} f_i \overline{g_i} \binom{d}{i}^{-1}; \tag{A.4.6}$$

- Canonical Hermitian inner product:

$$\langle f, g \rangle_{\mathbb{C}^{d+1}} := \sum_{i=0}^{d} f_i \overline{g_i}. \tag{A.4.7}$$

The solution variety is given by $\mathcal{V} = \{(f, z) \in \mathcal{P}_d \times \mathbb{C} : \, f(z) = 0\}$, and $\Sigma' = \{(f, z) \in \mathcal{V} : \, f'(z) = 0\}$. Thus, by implicit differentiation,

$$D\mathscr{S}(f)(\dot{f}) = -\left(f'(\zeta)\right)^{-1} \dot{f}(\zeta).$$

We denote by μ_W and $\mu_{\mathbb{C}^{d+1}}$ the condition numbers with respect to the Weyl and Hermitian inner product. The reader may check that

$$\mu_W(f,\zeta) = \frac{(1+|\zeta|^2)^{d/2}}{|f'(\zeta)|} \quad \text{and} \quad \mu_{\mathbb{C}^{d+1}}(f,\zeta) = \frac{\sqrt{\sum_{i=0}^{d} |\zeta|^{2i}}}{|f'(\zeta)|},$$

(for a proof see Blum *et al.* [1998], p. 228). From *Theorem 6*, we get:

$$\mu_{W\,st}(f,\zeta) = \frac{1}{\sqrt{2(d+1)}}\mu_W(f,\zeta), \qquad \mu_{\mathbb{C}^{d+1}\,st}(f,\zeta) = \frac{1}{\sqrt{2(d+1)}}\mu_{\mathbb{C}^{d+1}}(f,\zeta).$$

A.4.5 Finding Roots Problem II: Systems of Polynomial Equations

We now study the case of complex homogeneous polynomial systems. Let $\mathcal{H}_d$ be the space of homogeneous polynomials in $n+1$ complex variables of degree $d \in \mathbb{N} \setminus \{0\}$. We consider $\mathcal{H}_d$ with the Hermitian inner product $\langle \cdot, \cdot \rangle_d$, namely, the homogeneous analogous of the Weyl structure defined above (see Chapter 12 of Blum *et al.* [1998] for details).

Fix $d_1, \ldots, d_n \in \mathbb{N} \setminus \{0\}$ and let $\mathcal{H}_{(d)} = \mathcal{H}_{d_1} \times \cdots \times \mathcal{H}_{d_n}$ be the vector space of polynomial systems $f : \mathbb{C}^{n+1} \to \mathbb{C}^n$, $f = (f_1, \ldots, f_n)$, where $f_i \in \mathcal{H}_{d_i}$. The space $\mathcal{H}_{(d)}$ is naturally endowed with the Hermitian inner product $\langle f, g \rangle_W = \sum_{i=1}^{n} \langle f_i, g_i \rangle_{d_i}$.

Let $\mathcal{X} = \mathbb{P}(\mathcal{H}_{(d)})$ and $\mathcal{Y} = \mathbb{P}(\mathbb{C}^{n+1})$, then the solution variety is given by $\mathcal{V} = \{(f,\zeta) \in \mathbb{P}(\mathcal{H}_{(d)}) \times \mathbb{P}(\mathbb{C}^{n+1}) : f(\zeta) = 0\}$, and $\Sigma' = \{(f,\zeta) \in \mathcal{V} : Df(\zeta)|_{\zeta^\perp} \text{ is singular}\}$.

We denote by $N = \sum_{i=1}^{n} \binom{d_i+n}{n} - 1$ the complex dimension of $\mathcal{X}$. We may think of $2N$ as the size of the input.

Then, for $(f,\zeta) \in \mathcal{V} \setminus \Sigma'$, we have

$$D\mathscr{S}(f)\dot{f} = -\left(Df(\zeta)|_{\zeta^\perp}\right)^{-1} \dot{f}(\zeta),$$

and the condition number is

$$\mu_W(f,\zeta) = \left\| \left(Df(\zeta)|_{\zeta^\perp}\right)^{-1} \right\|,$$

where some norm 1 affine representatives of f and ζ have been chosen (cf. Blum *et al.* [1998]).

For the complexity analysis of path-following methods it is convenient to consider the *normalized condition number* defined by:

$$\mu_{norm}(f,\zeta) = \left\| \left(Df(\zeta)|_{\zeta^\perp}\right)^{-1} \cdot \mathrm{Diag}(d_1^{1/2}, \ldots, d_n^{1/2}) \right\|,$$

where $\mathrm{Diag}(d_1^{1/2}, \ldots, d_n^{1/2})$ denotes the diagonal matrix with entries $d_1^{1/2}, \ldots, d_n^{1/2}$. (Notice that μ_{norm} is the usual condition number for the slightly modified Hermitian inner product in $\mathcal{H}_{(d)}$ given by $\langle f, g\rangle_{norm} = \sum_{i=1}^n \frac{1}{d_i}\langle f_i, g_i\rangle_{d_i}$.)

Associated with μ_{norm}, we consider

$$\mu_{norm}(f)^2 := \frac{1}{\mathcal{D}} \sum_{\{\zeta\colon f(\zeta)=0\}} \mu_{norm}(f,\zeta)^2, \tag{A.4.8}$$

where $\mathcal{D} = d_1 \cdots d_n$ is the number of projective solutions of a generic system.

The expected value of $\mu^2_{norm}(f)$ is an essential ingredient in the complexity analysis of path-following methods (cf. Shub & Smale [1996], Beltrán & Pardo [2011], and recently Bürgisser & Cucker [2011]). In Beltrán & Pardo [2011] the authors proved that

$$\mathbb{E}_f \left[\mu_{norm}(f)^2\right] \leq 8nN, \tag{A.4.9}$$

where f is chosen at random with the Weyl distribution.

The relation between complexity theory and the stochastic condition number is not clear yet. However, it is interesting to study the expected value of the μ_{st}-analogue of equation (A.4.8), namely

$$\mu_{norm\,st}(f)^2 := \frac{1}{\mathcal{D}} \sum_{\{\zeta\colon f(\zeta)=0\}} \mu_{norm\,st}(f,\zeta)^2.$$

Here $\mu_{norm\,st}(f,\zeta)$ is the stochastic condition number for the modified condition operator, given by

$$\dot{f} \mapsto \left(Df(\zeta)|_{\zeta^\perp}\right)^{-1} \cdot \mathrm{Diag}(d_1^{1/2}, \ldots, d_n^{1/2}) \cdot \dot{f}(\zeta).$$

(Notice that, $\mu_{norm\,st}(f,\zeta)$ is the stochastic condition number for the modified

Hermitian inner product in $\mathcal{H}_{(d)}$ given by $\langle \cdot, \cdot \rangle_{norm}$).

From our *Theorem 6* we get,

$$\mu_{norm\,st}(f, \zeta) \leq \frac{\mu_{norm}(f, \zeta)}{\sqrt{N/n}}, \qquad \mathbb{E}_f \left[\mu_{norm\,st}(f))^2 \right] \leq 8\, n^2.$$

Note that the last bound depends on the number of unknowns n, and not on the size of the input $N \gg n$.

Part II

Random System of Equations

Chapter 4

Real Random Systems of Polynomials

In this chapter, following Armentano [2011b], we review some recent results concerning the expected number of real roots of random systems of polynomial equations. We begin giving an outline on Rice formulas for random fields. In the case of polynomial random fields we show the relation of Rice formulas with other technics to study the average number of solutions. At the end of this chapter we recall some known results about the undetermined case, that is, when the random system of equations has less equations than unknowns.

4.1 Introduction

Let us consider a system of m polynomial equations in m unknowns over $\mathbb{R}$,

$$f_i(x) := \sum_{\|j\| \leq d_i} a_j^{(i)} x^j \qquad (i = 1, \ldots, m). \tag{4.1.1}$$

The notation in (4.1.1) is the following: $x := (x_1, \ldots, x_m)$ denotes a point in $\mathbb{R}^m$, $j := (j_1, \ldots, j_m)$ a multi-index of non-negative integers, $\|j\| = \sum_{h=1}^{m} j_h$, $x^j = x^{j_1} \cdots x^{j_m}$, $a_j^{(i)} = a_{j_1,\ldots,j_m}^{(i)}$, and d_i is the degree of the polynomial f_i.

We are interested in the solutions of the system of equations

$$f_i(x) = 0, \qquad (i = 1, \ldots, m),$$

lying in some subset V of $\mathbb{R}^m$. We denote by $N^f(V)$ that number, and $N^f := N^f(\mathbb{R}^m)$

If we choose the coefficients $\{a_j^{(i)}\}$ at random , then $N^f(V)$ becomes a random variable.

The study of the expectation of the number of real roots of a random polynomial started in the thirties with the work of Bloch & Pólya [1931]. Further investigations were made by Littlewood & Offord [1938]. However, the first sharp result is due to Kac [1943; 1949], who gives the asymptotic value

$$\mathbb{E}\left(N^f(\mathbb{R})\right) \approx \frac{2}{\pi}\log d, \quad \text{as} \quad d \to +\infty,$$

when the coefficients of the degree d univariate polynomial f are Gaussian centered independent random variables $N(0,1)$ (see the book by Bharucha-Reid & Sambandham [1986]).

The first important result in the study of real roots of random system of polynomial equations is due to Shub & Smale [1993b], where the authors computed the expectation of $N^f(\mathbb{R}^m)$ when the coefficients are Gaussian centered independent random variables having variances:

$$\mathbb{E}\left[(a_j^{(i)})^2\right] = \frac{d_i!}{j_1!\cdots j_m!\,(d_i - \|j\|)!}. \tag{4.1.2}$$

Their result was

$$\mathbb{E}\left(N^f(\mathbb{R}^m)\right) = \sqrt{d_1\cdots d_m}, \tag{4.1.3}$$

that is, the square root of the Bézout number associated to the system. The proof in Shub & Smale [1993b] is based on a double fibration manipulation of the co-area formula (see formula (4.3.3) below).

The probability law of the Shub-Smale distribution has the simplifying property of being invariant under the action of the orthogonal group in $\mathbb{R}^m$. In Kostlan [2002] one can find the classification of all Gaussian probability distributions over the coefficients with this geometric invariant property.

Azaïs & Wschebor [2005] gave a new and deep insight to this problem, introducing the Rice formula for this problem. In Azaïs & Wschebor [2005], Rice formula allows them to extend the Shub-Smale result to other probability distributions over the coefficients. A general formula for $\mathbb{E}(N^f(V))$ when the random functions $f_i\,(i = 1, \ldots, m)$ are stochastically independent and their law is centered and invariant under the orthogonal group on $\mathbb{R}^m$ can be found in Azaïs & Wschebor [2005]. This includes Shub-Smale theorem as a special case.

Morever, Rice formula appears to be the instrument to consider a major problem in the subject which is to find the asymptotic distribution of $N^f(V)$ (under some normalization). The only published results of which the author is aware concern asymptotic variances as $m \to +\infty$. (See Wschebor [2008] for a detailed description in this direction).

4.2 Rice Formulas

We start this section giving an outline on Rice formulas for random fields. After that we will focus on polynomial random fields. This case is much simpler than the general theory of Rice formulas for random fields. At the end we will give an heuristic of Rice formula for polynomial random fields. In *Appendix B* we give a brief exposition of the main concepts about probability theory and stochastic processes used in this dissertation.

Let $U \subset \mathbb{R}^m$ be a Borel subset, and let $Z : U \to \mathbb{R}^m$ be a random field, that is, a collection $\{Z(x) : x \in U\}$ of random vectors defined on some probability space $(\Omega, \mathcal{A}, P)$.

Assume that the trajectories of the random field Z are regular. Given a value $u \in \mathbb{R}^m$, we denote $N_u^Z(U)$ the number of roots of $Z(x) = u$ lying in the subset U. Then, $N_u^Z(U) : \Omega \to \mathbb{N} \cup \{+\infty\}$ is a random variable. Rice formulas allow one to express the kth-moment of $N_u^Z(U)$ by an integral over U^k of a function that depends on the joint distribution of the process and its derivative. (See Azaïs & Wschebor [2009]).

More precisely, we have:

Theorem 7 (Rice Formula for the Expectation)**.** *Let $Z : U \to \mathbb{R}^m$ be a random field, $U \subset \mathbb{R}^m$ be an open set, and let $u \in \mathbb{R}^m$. Assume that*

1. *Z is Gaussian;*
2. *Almost surely the trajectories $x \mapsto Z(x)$ are C^1;*
3. *For each $x \in U$, $Z(x)$ has non-degenerate distribution, that is Var$(Z(x))$ is positive definite;*
4. *The event $\{\exists x \in U : Z(x) = u, \det(DZ(x)) = 0\}$ has probability zero.*

Then, one has

$$\mathbb{E}(N_u^Z(U)) = \int_U \mathbb{E}\left(|\det(DZ(x)|\,\middle|\, Z(x) = u\right)\, p_{Z(x)}(u)\, dx. \tag{4.2.1}$$

Here, $p_{Z(x)}(u)$ is the density of the random variable $Z(x)$ at u, and $\mathbb{E}(\xi|\eta = y)$ is the conditional expectation of ξ given the value of η at u (see *Appendix B.2*).

Theorem 8 (Rice Formula for the kth-moment)**.** *Let $k \geq 2$ be an integer. Assume the same hypotheses as in Theorem 7 except that 3. is repalced by*

3'. *For $x_1, \dots, x_k \in U$ distinct values, the distribution of $(Z(x_1), \dots, Z(x_k))$ does not degenerate in $(\mathbb{R}^m)^k$.*

Then, one has

$$\begin{aligned} &\mathbb{E}\big(N_u^Z(U)\,(N_u^Z(U) - 1) \cdots (N_u^Z(U) - k + 1)\big) = \qquad (4.2.2)\\ &= \int_{U^k} \mathbb{E}\left(\prod_{i=1}^{k} |\det DZ(x_i)|\,\middle|\, Z(x_1) = \dots Z(x_k) = u\right) \cdot \\ &\qquad\qquad \cdot\, p_{(Z(x_1),\dots,Z(x_n))}(u, \dots, u)\, dx_1 \dots dx_k. \end{aligned}$$

Theorem 9 (Expected Number of Weighted Roots)**.** *Let Z be a random field that verifies the hypotheses of Theorem 7. Moreover let $g : C(U, \mathbb{R}^m) \times U \to \mathbb{R}$ be a bounded function which is continuous when one puts on $C(U, \mathbb{R}^m)$ the topology of uniform convergence on compact sets. Then, for each compact set $I \subset U$, one has*

$$\mathbb{E}\Big(\sum_{x \in I:\, Z(x) = u} g(Z, x)\Big) = \int_I \mathbb{E}\left(g(Z, x)\, |\det(DZ(x))|\,\middle|\, Z(x) = u\right)\, p_{Z(x)}(u)\, dx. \tag{4.2.3}$$

For the proof of *Theorem 7*, *Theorem 8* and *Theorem 9* see Azaïs & Wschebor [2009].

More generally, let $U \subset \mathbb{R}^m$ be an open set, and let $f : U \to \mathbb{R}^k$ be a smooth function, where $m > k$. If $u \in \mathbb{R}^k$ is a regular value of f, then, $f^{-1}(u)$ is a smooth manifold of dimension $m-k$. Let us denote by λ_{m-k} the $m-k$ geometric measure.

Theorem 10 (Rice Formula for the Expectation of the Geometric Measure)**.** *Let $Z : U \to \mathbb{R}^k$ be a random field, $U \subset \mathbb{R}^m$ be an open set, and let $u \in \mathbb{R}^k$, $(m \geq k)$. Assume that*

1. *Z is Gaussian;*
2. *Almost surely the trajectories $x \mapsto Z(x)$ are C^1;*
3. *For each $x \in U$, $Z(x)$ has non-degenerate distribution, that is Var$(Z(x))$ is positive definite;*
4. *The event $\{\exists x \in U : \, Z(x) = u, \, \text{rank}(DZ(x)) < k\}$ has probability zero.*

Then, one has

$$\mathbb{E}(\lambda_{m-k}(f^{-1}(u) \cap U)) = \int_U \mathbb{E}\left(|\det[(DZ(x)) \cdot (DZ(x))^T]|^{1/2} \,\middle|\, Z(x) = u\right) \, p_{Z(x)}(u) \, dx, \tag{4.2.4}$$

where $\cdot^T$ means the transpose.

Remark:

- If instead of an open subset $U \subset \mathbb{R}^m$, the set U where we count the number of solutions $Z(x) = u$ is an open subset of a differential manifold, the same formulas hold replacing the Lebesgue measure by the geometric measure of the manifold and the derivative of the random field by the derivative along the manifold.
- In general, condition 4. of *Theorem 7* may be difficult to prove. However in the case of random polynomial systems it holds. Note that this condition is a "Sard" type condition.

- *Theorem 9* is a particular case of *Theorem 6.4* of Azaïs & Wschebor [2009].

In this thesis we will restrict ourselves to the particular case of random fields, namely, polynomial random fields. This is our next tasks.

4.3 Polynomial Random Fields

Following the notation in *Section 4.1*, when we randomize the coefficients $\{a_j^{(i)}\}$ in some probability space $(\Omega, \mathcal{A}, P)$, the polynomial system f becomes a random field. Let us denote by $Z : \Omega \times \mathbb{R}^m \to \mathbb{R}^m$ that random field, that is

$$Z_i(\omega, x) = \sum_{\|j\| \leq d_i} a_j^{(i)}(\omega) x^j, \qquad (i = 1, \dots, m). \tag{4.3.1}$$

Here the situation is much simpler as compared with the general theory of stochastic processes. The main reason for that is that the set of functions

$$\{Z(\omega, \cdot) : \mathbb{R}^m \to \mathbb{R}^m, \quad \omega \in \Omega\},$$

lives in a finite dimensional subspace of the infinite dimensional space $\mathcal{F}(\mathbb{R}^m, \mathbb{R}^m)$ of functions from $\mathbb{R}^m$ to $\mathbb{R}^m$. Let us be more precise.

For $(d) = (d_1, \dots, d_n)$, let $\mathscr{P}_{(d)} = \mathscr{P}_{d_1} \times \cdots \times \mathscr{P}_{d_m}$ be the space of m polynomial equations in m real variables, where $\mathscr{P}_d$ stands for the vector space of degree d polynomials in m real variables. Note that $\mathscr{P}_{(d)} \subset \mathcal{F}(\mathbb{R}^m, \mathbb{R}^m)$ and can be identified with the finite dimensional vector space $\mathbb{R}^{\dim(\mathscr{P}_{(d)})}$. In the next lines we will write $\mathscr{P}_{(d)}$ but we may think on this identification.

Fixing $\omega \in \Omega$, we get that $Z(\omega, \cdot) \in \mathscr{P}_{(d)}$. Then, we have the natural map

$$\xi : \Omega \to \mathscr{P}_{(d)}, \qquad \xi(\omega) = Z(\omega, \cdot) = (a_j^{(i)}(\omega))_{\substack{\|j\| \leq d_i \\ i=1,\dots,m}}. \tag{4.3.2}$$

Therefore ξ is a random vector on $\mathscr{P}_{(d)}$, that is, a measurable function from $(\Omega, \mathcal{A})$ to $(\mathscr{P}_{(d)}, \mathcal{B})$, where $\mathcal{B}$ is the Borel σ-algebra of $\mathscr{P}_{(d)}$. Then, ξ induces a probability measure on $(\mathscr{P}_{(d)}, \mathcal{B}, \nu)$, namely, the push forward measure

$$\nu(B) = P(\xi^{-1}(B)) = P(\{\omega \in \Omega : \, (a_j^{(i)}(\omega))_{\substack{\|j\| \leq d_i \\ i=1,\dots,m}} \in B\}),$$

for all $B \in \mathcal{B}$.

Canonical Process

In this way we can define a new random field defined on the probability space $(\mathscr{P}_{(d)}, \mathcal{B}, \nu)$, as

$$Z : \mathscr{P}_{(d)} \times \mathbb{R}^m \to \mathbb{R}^m, \qquad Z(f, x) = f(x).$$

This random field is known as the *canonical process* (see *Appendix B*). (Note that Z is just the evaluation map.)

In this case the associated map ξ given in (4.3.2), is the identity map, that is $Z(f, \cdot) = f$. Then this new random field induces the same probability measure on the space $\mathscr{P}_{(d)}$ as the random field (4.3.1), and therefore they can be seen as the same process.

These observations lead us to give a geometric structure to the probability space $(\Omega, \mathcal{A}, P)$, just defining $\Omega = \mathscr{P}_{(d)}$, $\mathcal{A} = \mathcal{B}$, and $\mathbb{P} = \nu$. This is the main purpose of next section, trying to relate Rice formulas with known formulas in geometric integrations, as the co-area formula.

However, we alert the reader that this is not necessary to study a stochastic processes. One can just work with the probability space $(\Omega, \mathcal{A}, P)$ with no more structure than the given one.

Remark 4.3.1. Let $Z : \Omega \times \mathbb{R}^m \to \mathbb{R}^m$ be a random field. In the general theory of stochastic processes the trajectories $Z(\omega, \cdot)$ lies, in general, in the infinite dimensional space $\mathcal{F}(\mathbb{R}^m, \mathbb{R}^m)$. Then, the main problem is how to introduce in $\mathcal{F}(\mathbb{R}^m, \mathbb{R}^m)$ a σ-algebra and a measure such that the map $\xi : \Omega \to \mathcal{F}(\mathbb{R}^m, \mathbb{R}^m)$, given by $\xi(\omega) = Z(\omega, \cdot)$, is a measurable function, and the measure on $\mathcal{F}(\mathbb{R}^m, \mathbb{R}^m)$ is just the push forward measure by ξ. This is a non trivial issue, and under mild conditions on the random field, this construction is possible. This construction was made by Kolmogorov and it is known as *Kolmogorov extension Theorem* (see [Azaïs & Wschebor, 2009, Theorem 1.1, page 12]) .

4.3.1 Rice Formula Heuristic

In this section we show the relation of Rice Formula and other technics used in integral geometry.

Let $\Omega = \mathscr{P}_{(d)}$, with the Borel σ-algebra and let ν be any probability measure on $\mathscr{P}_{(d)}$. Let Z be the random field $Z : \mathscr{P}_{(d)} \times \mathbb{R}^m \to \mathbb{R}^m$ given by the evaluation map, that is $Z(f, x) = f(x)$.

Note that, for a fixed $x \in \mathbb{R}^m$, $Z(\cdot, x) : \mathscr{P}_{(d)} \to \mathbb{R}^m$ is the function $f \mapsto f(x)$. Moreover, for a fixed $f \in \mathscr{P}_{(d)}$, we have $Z(f, \cdot) = f(\cdot)$.

Moreover, in this case, the random variable $N^Z(U)$ is given by

$$\begin{aligned} N^Z(U) : &\mathscr{P}_{(d)} \to \mathbb{N} \cup \{+\infty\}, \\ &f \mapsto \#_U f^{-1}(0) \end{aligned}$$

that is, the number of solutions of $f(x) = 0$, lying in the subset $U \subset \mathbb{R}^m$. Therefore, we can write

$$\mathbb{E}(N^Z(U)) = \int_{f \in \mathscr{P}_{(d)}} \#_U f^{-1}(0)\, d\nu.$$

Assume that the random field Z satisfy the condition of Rice formula in *Theorem 7*.

Then applying Rice Formula (4.2.1) we get

$$\mathbb{E}(N^Z(U)) = \int_U \mathbb{E}\left(|\det(DZ(x)|\,\big|\, Z(x) = 0\right)\, p_{Z(x)}(0)\, dx.$$

For a fixed $x \in \mathbb{R}^m$, the event $\{Z(x) = 0\}$ is the subset of $\mathscr{P}_{(d)}$ given by $\{f \in \mathscr{P}_{(d)} :\, Z(f, x) = 0\}$, that is $\{Z(x) = 0\}$ is the vector subspace of $\mathscr{P}_{(d)}$ given by

$$\mathcal{V}_x = \{f \in \mathscr{P}_{(d)} :\, f(x) = 0\}.$$

Note that $|\det(DZ(x))|$ at $f \in \mathscr{P}_{(d)}$ is $|\det(Df(x))|$. Therefore, the conditional expectation $\mathbb{E}\left(|\det(DZ(x))|\,\big|\, Z(x) = 0\right)$ is the integral in the fiber $\mathcal{V}_x$ of the function $|\det(Df(x))|$ with respect to the conditional probability measure ν_x. Note that $\nu_x(\mathcal{V}_x) = 1$.

The density $p_{Z(x)}(y)$, for some $y \in \mathbb{R}^m$, is associated to the measure of the fiber. More precisely, let $B_\varepsilon(y) \subset \mathbb{R}^m$ be the Euclidean ball of center y and radius $\varepsilon > 0$, then

$$\int_{B_\varepsilon(y)} p_{Z(x)}(z)\, dz = \nu(\{f \in \mathscr{P}_{(d)} : f(x) \in B_\varepsilon(y)\}).$$

Therefore, since $Z(x)$ is a non-degenerate Gaussian random vector, we get

$$p_{Z(x)}(y) = \lim_{\varepsilon \downarrow 0} \frac{\nu(\{f \in \mathscr{P}_{(d)} : f(x) \in B_\varepsilon(y)\})}{\lambda_m(B_\varepsilon(y))}.$$

Then, we can rewrite Rice formula as

$$\int_{f \in \mathscr{P}_{(d)}} \#_U f^{-1}(0)\, d\nu = \int_{x \in U} \left(\int_{f \in \mathcal{V}_x} |\det(Df(x)|\, d\mathcal{V}_x \right) dx, \tag{4.3.3}$$

where $d\mathcal{V}_x$ is the (non-normalized) measure $p_{Z(x)}(0) \cdot d\nu_x$.

Formula (4.3.3) is the type of formula used by Shub and Smale to study the number of solutions of random system of equations. They arrive to this type of formula by a double fibration of the co-area formula (see [Blum *et al.*, 1998, Theorem 5, page 243]).

Remark 4.3.2. All the preceding observations applies *mutatis mutandis* to the space $\mathcal{H}_{(d)}$ of homogeneous polynomial systems.

Remark 4.3.3. Note that, when the measure ν_{d_i} is a Gaussian measure on $\mathscr{P}_{d_i}$, then, the covariance $\Gamma_i : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$ of the stochastic process Z_i, is given by

$$\Gamma_i(x,y) = \mathbb{E}(Z_i(x)Z_i(y)) = \int_{\omega \in \Omega} Z_i(\omega, x) Z_i(\omega, y)\, dP(w) = \int_{g \in \mathscr{P}_{d_i}} g(x)g(y)\, d\nu_{d_i}.$$

That is,

$$\Gamma_i(x,y) = \langle K_x, K_y \rangle_{L^2},$$

is the $L^2(\mathscr{P}_{d_i}, \mathcal{B}, \nu_{d_i})$ inner product of the evaluation map functions $K_z : \mathscr{P}_{d_i} \to \mathbb{R}$ given by $K_z(g) = g(z)$, for all $z \in \mathbb{R}^m$.

In the particular case that ν_{d_i} is the measure defined by the Weyl inner product: for $j = (j_1, \dots, j_m) \in \mathbb{N}^m$, $\|j\| = d_i$ the monomial $x^j = x_1^{j_1} \cdots x_m^{j_m}$, the Weyl inner product makes $\langle x^j, x^{j'} \rangle = 0$, for $j \neq j'$ and

$$\langle x^j, x^j \rangle = \binom{d_i}{j}^{-1},$$

is not difficult to see that $\langle K_x, K_y \rangle_{L^2} = (1 + \langle x, y \rangle)^{d_i}$, and therefore $\Gamma_i(x, y) = (1 + \langle x, y \rangle)^{d_i}$.

4.4 Shub-Smale Distribution

Let us consider a system of m polynomial equations in m unknowns over $\mathbb{R}$,

$$f_i(x) := \sum_{\|j\| \le d_i} a_j^{(i)} x^j, \qquad (i = 1, \dots, m). \tag{4.4.1}$$

We say that this system of equation has the *Shub-Smale distribution* when the coefficients are Gaussian centered independent random variables having variances

$$\mathbb{E}\left[(a_j^{(i)})^2\right] = \frac{d_i!}{j_1! \cdots j_m! \, (d_i - \|j\|)!}. \tag{4.4.2}$$

Theorem 11 (Shub-Smale)**.** *Let f be the system of equations (4.4.1) with the Shub-Smale distribution. Then*

$$\mathbb{E}(N^f) = \sqrt{d_1 \cdots d_m}.$$

Proof. Let us homogenize the system of polynomials, that is, let $F : \mathbb{R}^{m+1} \to \mathbb{R}^m$, where

$$F_i(x_0, \dots, x_m) := \sum_{\|j\| = d_i} a_j^{(i)} x_0^{j_0} x_1^{j_1} \cdots x_m^{j_m}, \qquad (i = 1, \dots, m).$$

Note that

$$N^f = \frac{1}{2} N^F(S^m). \tag{4.4.3}$$

Claim I: The random polynomials F_i are independent, Gaussian, centered, with covariance function $\Gamma_i(x, y) = \langle x, y \rangle^d$:
This claims follows immediately from the definition of the Shub-Smale distribution.

Claim II: The derivative of F along S^m at $x \in S^m$, $DF(x)|_{x^\perp}$, is independent of $F(x)$:
Since $\mathbb{E}(F_i(x)^2) = 1$ for all $x \in S^m$, the claims follows differentiating under expectation sign.

Claim III: The law of the random field F is invariant under the action of the orthogonal group of $\mathbb{R}^{m+1}$:
This follows from *Claim I*, since the covariance of the process is invariant under this group.

By Rice Formula we get

$$\mathbb{E}(N^F(S^m)) = \int_{S^m} \mathbb{E}(|\det DF(x)|_{x^\perp}| \,\big|\, F(x) = 0) p_{F(x)}(0)\, dS^m(x),$$

where dS^m is the geometric measure of S^m. Note that $F(x)$ is a standard Gaussian random vector in $\mathbb{R}^m$, therefore $p_{F(x)}(0) = (2\pi)^{-m/2}$. Then

$$\begin{aligned}
\mathbb{E}(N^F(S^m)) &= \frac{1}{(2\pi)^{m/2}} \int_{S^m} \mathbb{E}(|\det DF(x)|_{x^\perp}| \,\big|\, F(x) = 0)\, dS^m(x) \\
&= \frac{1}{(2\pi)^{m/2}} \int_{S^m} \mathbb{E}(|\det DF(x)|_{x^\perp}|)\, dS^m(x) \\
&= \frac{1}{(2\pi)^{m/2}} \mathrm{vol}(S^m) \mathbb{E}(|\det DF(e_0)|_{e_0^\perp}|),
\end{aligned}$$

where the successive equalities follows from *Claim II* and *Claim III*.

Differentiating Γ_i under the expectation sign, we obtain

$$\frac{\partial}{\partial x_\ell} \frac{\partial}{\partial x_{\ell'}} \Gamma_i(x, y)\big|_{x=y=e_0} = \mathbb{E}(\frac{\partial F_i}{\partial x_\ell}(e_0) \frac{\partial F_i}{\partial x_{\ell'}}(e_0)) = d_i \delta_{\ell \ell'}. \tag{4.4.4}$$

Therefore,

$$DF(e_0)|_{e_0^\perp} = \Delta(\sqrt{d_i}) \cdot G,$$

where $\Delta(d_i)$ is the diagonal matrix whose ith entry is d_i and G is an $m \times m$ standard Gaussian matrix, that is, a $m \times m$ matrix with i.i.d standard Gaussian entries.

Hence

$$\mathbb{E}(N^F(S^m)) = \frac{1}{(2\pi)^{m/2}} \mathrm{vol}(S^m) \sqrt{d_1 \cdots d_m} \mathbb{E}(|\det G|). \tag{4.4.5}$$

Thereby, we reduce the problem of finding the expected value of the number of roots of a random system to a problem of random matrices, namely, compute $\mathbb{E}(|\det G|)$ for a standard Gaussian matrix.

The computation of $\mathbb{E}(|\det G|)$ is quite standard and should be interpreted as the expected value of the volume of a random parallelepiped. For a proof see the book by Azaïs & Wschebor [2009].

One has

$$\mathbb{E}(|\det G|) = \frac{1}{\sqrt{2\pi}} 2^{(m+1)/2}\Gamma((m+1)/2). \tag{4.4.6}$$

The proof follows from (4.4.3), (4.4.5) and (4.4.6). □

Remark 4.4.1. The given proof of *Theorem 11* is due to Jean-Marc Azaïs and Mario Wschebor and is included in Azaïs & Wschebor [2009]. This proof shows the power of Rice formula to study this kind of problems. In many situations we have similar conditions and we can proceed as in the proof of *Theorem 11*. Roughly, the conditions are: invariance of the law under certain group of motions, and the independence of the condition in the conditional expectation. In these cases the problem always reduce to a problem of random matrices. See for example *Theorem 16*, *Theorem 15* or *Theorem 18*. Moreover, if instead of counting roots we consider the problem of computing weighted roots, where the function we ponderate on the roots is a function of the derivative of the process, then we can proceed as we mention before, reducing our problem to a problem of random matrices. That is the case, for example, when we ponderate the condition number at each root.

4.5 Non-centered Systems

The aim of this section is to remove the hypothesis that the coefficients have zero expectation.

One way to look at this problem is to start with a non-random system of equations (the “signal”)

$$P_i(x) = 0, \qquad (i = 1, \dots, m), \tag{4.5.1}$$

perturb it with a polynomial noise $X_i(x)$ $(i = 1, \dots, m)$, that is, consider

$$P_i(x) + X_i(x) = 0, \qquad (i = 1, \dots, m),$$

and ask what one can say about the number of roots of the new system, or, how much the noise modifies the number of roots of the deterministic part. (For short, we denote $N^f = N^f(\mathbb{R}^m)$).

Roughly speaking, we prove in *Theorem 12* that if the relation signal over noise is neither too big nor too small, in a sense that will be made precise later on this chapter, there exist positive constants C, θ, where $0 < \theta < 1$, such that

$$\mathbb{E}(N^{P+X}) \leq C\,\theta^m \mathbb{E}(N^X). \tag{4.5.2}$$

Inequality (4.5.2) becomes of interest if the starting non-random system (4.5.1) has a large number of roots, possibly infinite, and m is large. In this situation, the effect of adding polynomial noise is a reduction at a geometric rate of the expected number of roots, as compared to the centered case in which all the P_i's are identically zero.

For simplicity we assume that the polynomial noise X has the Shub-Smale distribution (4.4.2). However, one should keep in mind that the result can be extended to other orthogonally invariant distributions (cf. Armentano & Wschebor [2009]).

Before the statement of *Theorem 12* below, we need to introduce some additional notations.

In this simplified situation, one only needs hypotheses concerning the relation between the signal P and the Shub-Smale noise X, which roughly speaking should neither be too small nor too big.

Since X has the Shub-Smale distribution, from (4.4.2) we get

$$\operatorname{Var}(X_i(x)) = (1 + \|x\|^2)^{d_i}, \quad \forall x \in \mathbb{R}^m, \qquad (i = 1, \ldots, m),$$

(see *Remark 4.3.3*).

Define

$$H(P_i) := \sup_{x \in \mathbb{R}^m} \left\{ (1 + \|x\|) \cdot \left\| \nabla \left(\frac{P_i}{(1 + \|x\|^2)^{d_i/2}} \right)(x) \right\| \right\},$$
$$K(P_i) := \sup_{x \in \mathbb{R}^m \setminus \{0\}} \left\{ (1 + \|x\|^2) \cdot \left| \frac{\partial}{\partial \rho} \left(\frac{P_i}{(1 + \|x\|^2)^{d_i/2}} \right)(x) \right| \right\},$$

for $i = 1, \ldots, m$, where $\|\cdot\|$ is the Euclidean norm, and $\frac{\partial}{\partial \rho}$ denotes the derivative in the direction defined by $\frac{x}{\|x\|}$, at each point $x \neq 0$.

For $r > 0$, put:

$$L(P_i, r) := \inf_{\|x\| \geq r} \frac{P_i(x)^2}{(1 + \|x\|^2)^{d_i}} \quad (i = 1, \ldots, m).$$

One can check by means of elementary computations that for each P as above, one has

$$H(P) < \infty, \; K(P) < \infty.$$

With these notations, we introduce the following hypotheses on the systems as m grows:

H_1)

$$A_m = \frac{1}{m} \cdot \sum_{i=1}^{m} \frac{H^2(P_i)}{i} = \mathrm{o}(1) \quad \text{as } m \to +\infty \tag{4.5.3a}$$

$$B_m = \frac{1}{m} \cdot \sum_{i=1}^{m} \frac{K^2(P_i)}{i} = \mathrm{o}(1) \quad \text{as } m \to +\infty. \tag{4.5.3b}$$

H_2) There exist positive constants r_0, ℓ such that if $r \geq r_0$:

$$L(P_i, r) \geq \ell \quad \text{for all } i = 1, \ldots, m.$$

Theorem 12. *Under the hypotheses H_1) and H_2), one has*

$$\mathbb{E}(N^{P+X}) \leq C\, \theta^m \mathbb{E}(N^X), \tag{4.5.4}$$

where C, θ are positive constants, $0 < \theta < 1$.

Remarks on the statement of *Theorem 12*

- It is obvious that our problem does not depend on the order in which the equations

  $$P_i(x) + X_i(x) = 0 \quad (i = 1, \ldots, m)$$

  appear. However, conditions (4.5.3a) and (4.5.3b) in hypothesis H_3) do depend on the order. One can state them by saying that there exists an order $i = 1, \ldots, m$ on the equations, such that (4.5.3a) and (4.5.3b) hold true.

- Condition H_1) can be interpreted as a bound on the quotient signal over noise. In fact, it concerns the gradient of this quotient. In (4.5.3b) the radial derivative appears, which happens to decrease faster as $\|x\| \to \infty$ than the other components of the gradient.

  Clearly, if $H(P_i)$, $K(P_i)$ are bounded by fixed constants, (4.5.3a) and (4.5.3b) are verified. Also, some of them may grow as $m \to +\infty$ provided (4.5.3a) and (4.5.3b) remain satisfied.

- Hypothesis H_2) goes – in some sense – in the opposite direction: For large values of $\|x\|$ we need a lower bound of the relation signal over noise.

- A result of the type of *Theorem 12* can not be obtained without putting some restrictions on the relation signal over noise. In fact, consider the system

$$P_i(x) + \sigma X_i(x) = 0 \quad (i = 1, \ldots, m), \tag{4.5.5}$$

  where σ is a positive real parameter. If we let $\sigma \to +\infty$, the relation signal over noise tends to zero and the expected number of roots will tend to $\mathbb{E}(N^X)$. On the other hand, if $\sigma \downarrow 0$, $\mathbb{E}(N^X)$ can have different behaviours. For example, if P is a "regular" system, the expected value of the number of roots of (4.5.5) tends to the number of roots of $P_i(x) = 0$, $(i = 1, \ldots, m)$, which may be much bigger than $\mathbb{E}(N^X)$. In this case, the relation signal over noise tends to infinity.

- As it was mentioned before we can extend *Theorem 12* to other orthogonally invariant distributions. However, for the general version we need to add more hypotheses.

In the next paragraphs we are going to give two simple examples.

For the proof of *Theorem 12* and more examples with different noises see Armentano & Wschebor [2009].

4.5.1 Some Examples

We assume that the degrees d_i are uniformly bounded as m growth.

For the first example, let

$$P_i(x) = \|x\|^{d_i} - r^{d_i},$$

where d_i is even and r is positive and remains bounded as m varies. Then, one has:

$$\frac{\partial}{\partial\rho}\left(\frac{P_i}{(1+\|x\|^2)^{d_i/2}}\right)(x) = \frac{d_i\,\|x\|^{d_i-1} + d_i\,r^{d_i}\,\|x\|}{(1+\|x\|^2)^{\frac{d_i}{2}+1}} \leq \frac{d_i(1+r^{d_i})}{(1+\|x\|^2)^{3/2}}$$
$$\nabla\left(\frac{P_i}{(1+\|x\|^2)^{d_i/2}}\right)(x) = \frac{d_i\,\|x\|^{d_i-2} + d_i\,r^{d_i}}{(1+\|x\|^2)^{\frac{d_i}{2}+1}}\,x$$

which implies

$$\left\|\nabla\left(\frac{P_i}{(1+\|x\|^2)^{d_i/2}}\right)(x)\right\| \leq \frac{d_i(1+r^{d_i})}{(1+\|x\|^2)^{3/2}}.$$

Again, since the degrees $d_1,\ldots,d_m$ are bounded by a constant that does not depend on m, H_1) follows. H_2) also holds under the same hypothesis.

Notice that an interest in this choice of the P_i's lies in the fact that obviously the system $P_i(x) = 0$ $(i = 1,\ldots,m)$ has an infinite number of roots (all points in the sphere of radius r centered at the origin are solutions), but the expected number of roots of the perturbed system is geometrically smaller than the Shub–Smale expectation, when m is large.

Our second example is the following: Let T be a polynomial of degree d in one variable that has d distinct real roots. Define:

$$P_i(x_1,\ldots,x_m) = T(x_i) \quad (i = 1,\ldots,m).$$

One can easily check that the system verifies our hypotheses, so that there exist

C, θ positive constants, $0 < \theta < 1$ such that

$$\mathbb{E}(N^{P+X}) \leq C\,\theta^m d^{m/2},$$

where we have used the Shub–Smale formula when the degrees are all the same. On the other hand, it is clear that $N^P = d^m$ so that the diminishing effect of the noise on the number of roots can be observed. A number of variations of these examples for P can be constructed, but we will not pursue the subject here.

4.6 Bernstein Polynomial Systems

Up to now all probability measures were introduced in a particular basis, namely, the monomial basis $\{x^j\}_{\|j\| \leq d}$. However, in many situations, polynomial systems are expressed in different basis, such as, orthogonal polynomials, harmonic polynomials, Bernstein polynomials, etc. So, it is a natural question to ask: *What can be said about $N^f(V)$ when the randomization is performed in a different basis?*

For the case of random orthogonal polynomials see Bharucha-Reid & Sambandham [1986], and Edelman & Kostlan [1995] for random harmonic polynomials.

In this section following Armentano & Dedieu [2009] we give an answer to the average number of real roots of a random system of equations expresed in the Bernstein basis. Let us be more precise:

The Bernstein basis is given by:

$$b_{d,k}(x) = \binom{d}{k} x^k (1-x)^{d-k}, \qquad 0 \leq k \leq d,$$

in the case of univariate polynomials, and

$$b_{d,j}(x_1, \ldots, x_m) = \binom{d}{j} x_1^{j_1} \ldots x_m^{j_m} (1 - x_1 - \ldots - x_m)^{d-\|j\|}, \quad \|j\| \leq d,$$

for polynomials in m variables, where $j = (j_1, \ldots, j_m)$ is a multi-integer, and $\binom{d}{j}$ is the multinomial coefficient.

Let us consider the set of real polynomial systems in m variables,

$$f_i(x_1,\ldots,x_m) = \sum_{\|j\|\le d_i} a_j^{(i)} b_{d,j}(x_1,\ldots,x_m), \qquad (i=1,\ldots,m).$$

Take the coefficients $a_j^{(i)}$ to be independent Gaussian standard random variables.

Define

$$\tau : \mathbb{R}^m \to \mathbb{P}\left(\mathbb{R}^{m+1}\right)$$

by

$$\tau(x_1,\ldots,x_m) = [x_1,\ldots,x_m, 1-x_1-\ldots-x_m].$$

Here $\mathbb{P}\left(\mathbb{R}^{m+1}\right)$ is the projective space associated with $\mathbb{R}^{m+1}$, $[y]$ is the class of the vector $y \in \mathbb{R}^{m+1}$, $y \neq 0$, for the equivalence relation defining this projective space. The (unique) orthogonally invariant probability measure in $\mathbb{P}\left(\mathbb{R}^{m+1}\right)$ is denoted by λ_m.

With the above notation the following theorem holds:

Theorem 13. *1. For any Borel set V in $\mathbb{R}^m$ we have*

$$\mathbb{E}\left(N^f(V)\right) = \lambda_m(\tau(V))\sqrt{d_1\ldots d_m}.$$

In particular

2. $\mathbb{E}\left(N^f\right) = \sqrt{d_1\ldots d_m}$,

3. $\mathbb{E}\left(N^f(\Delta^m)\right) = \sqrt{d_1\ldots d_m}/2^m$, *where*

$$\Delta^m = \left\{x \in \mathbb{R}^m \ :\ x_i \ge 0 \text{ and } x_1+\ldots+x_m \le 1\right\},$$

4. When $m=1$, for any interval $I=[\alpha,\beta]\subset\mathbb{R}$, one has

$$\mathbb{E}\left(N^f(I)\right) = \frac{\sqrt{d}}{\pi}\left(\arctan(2\beta-1)-\arctan(2\alpha-1)\right).$$

The fourth assertion in *Theorem 13* is deduced from the first assertion but it also can be derived from Crofton's formula (see for example Edelman & Kostlan [1995]).

Let us denote by $\mathcal{H}_{(d)}$ the space of real homogeneous polynomial systems in $m+1$ variables, $F = (F_1, \ldots, F_m)$, where

$$F_i(x_1, \ldots, x_m, x_{m+1}) = \sum_{|j| \leq d_i} a_j^{(i)} x_1^{j_1} \ldots x_m^{j_m} x_{m+1}^{d_i - |j|}.$$

$(d) = (d_1, \ldots, d_m)$ denotes the vector of degrees, $d_i \geq 1$, and $\deg F_i = d_i$ for every i.

Assume that the coefficients $a_j^{(i)}$ are independent centered Gaussian variables with variance $\binom{d_i}{j}$. The real roots of such a system consist in lines through the origin in $\mathbb{R}^{m+1}$ which are identified to points in $\mathbb{P}\left(\mathbb{R}^{m+1}\right)$.

Theorem 14. *For any measurable set $\mathcal{B} \subset \mathbb{P}\left(\mathbb{R}^{m+1}\right)$ we have*

$$\mathbb{E}\left(N^F(\mathcal{B})\right) = \lambda_m(\mathcal{B})\sqrt{d_1 \ldots d_m}.$$

Proof of Theorem 14. For any measurable set $\mathcal{B} \subset \mathbb{P}\left(\mathbb{R}^{m+1}\right)$ let us define

$$\mu_n(\mathcal{B}) = \mathbb{E}\left(N^F(\mathcal{B})\right).$$

We see that μ_m is an orthogonally invariant measure in $\mathbb{P}\left(\mathbb{R}^{m+1}\right)$. Thus it is equal to λ_m up to a multiplicative factor. From *Theorem 11*, this factor is equal to $\sqrt{d_1 \ldots d_m}$. Therefore

$$\mathbb{E}\left(N^F(\mathcal{B})\right) = \lambda_m(\mathcal{B})\sqrt{d_1 \ldots d_m}.$$

□

Proof of Theorem 13. Let us prove the first item. For any measurable set $B \subset \mathbb{R}^m$ we have by *Theorem 14*

$$\lambda_m(\tau(B))\sqrt{d_1 \ldots d_m} = \mathbb{E}\left(N^F(\tau(B))\right) = \int_{\mathcal{H}_{(d)}} N^F(\tau(B))dF.$$

The map h which associates to $f \in \mathcal{P}_{(d)}$ the homogeneous system $F \in \mathcal{H}_{(d)}$ obtained in substituting x_{m+1} to the affine form $(1 - x_1 - \ldots - x_m)$ is an isometry between these two spaces so that

$$\int_{\mathcal{H}_{(d)}} N^F(\tau(B))\,dF = \int_{\mathcal{P}_{(d)}} N^{h(f)}(\tau(B))df.$$

Since $N^{h(f)}(\tau(B)) = N^f(B)$ this last integral is equal to $\int_{\mathscr{P}_{(d)}} N^f(B)\,df$.

To complete the proof of this theorem we notice that $\lambda_m(\tau(\mathbb{R}^m)) = 1$, $\lambda_m(\tau(S_m)) = 1/2^n$, and,

$$\lambda_1(\tau([\alpha,\beta])) = \frac{1}{\pi}\int_\alpha^\beta \frac{1}{t^2+(1-t)^2}\,dt = \frac{\arctan(2\beta-1)-\arctan(2\alpha-1)}{\pi},$$

which follows from the computation of the length of the path $\{\tau(t)\}_{t\in[\alpha,\beta]} \subset \mathbb{P}(\mathbb{R})$. □

4.6.1 Some Extensions: Random Equations with a Simple Answer

In this section we extend last result on Bernstein polynomial systems. We give a general formula to compute the expected number of roots of some random systems of equations.

Let $U \subset \mathbb{R}^m$ be an open subset, and let $\varphi_0,\ldots,\varphi_m : U \to \mathbb{R}$ be $(m+1)$ differentiable functions. Assume that, for every $x \in U$, the values $\varphi_i(x)$ do not vanish at the same time. Then we can define the map $\Lambda : U \to \mathbb{P}(\mathbb{R}^{m+1})$ by $\Lambda(x) = [\varphi_0(x),\ldots,\varphi_m(x)]$.

Let f be the system of m-equations in m real variables

$$f_i(x_1,\ldots,x_m) := \sum_{\|j\|=d_i} a_j^{(i)}\varphi_0(x)^{j_0}\cdots\varphi_m(x)^{j_m}, \qquad (i=1,\ldots,m), \tag{4.6.1}$$

where $x = (x_1,\ldots,x_m) \in U$.

We denote by $N^f(U)$ the number of roots of the system of equations $f_i(x) = 0$, $(i = 1,\ldots,m)$ lying in U. Then,

Theorem 15. *Let f be the system of equations (4.6.1), where the $\{a_j^{(i)}\}$ are independent Gaussian centered random variables with variance $\binom{d_i}{j}$. Then,*

$$\mathbb{E}\left[N^f(U)\right] = \frac{\sqrt{d_1 \cdots d_m}}{vol(\mathbb{P}(\mathbb{R}^{m+1}))} \int_{z \in \mathbb{P}(\mathbb{R}^{m+1})} \# \Lambda^{-1}(\{z\})\, dz.$$

where $\# \emptyset = 0$.

Proof. Let us denote by $F : \mathbb{R}^{m+1} \to \mathbb{R}^m$ the random field given by

$$F_i(z_0, \ldots, z_m) = \sum_{\|j\|=d_i} a_j^{(i)} z_0^{j_0} \cdots z_m^{j_m},$$

where $a_j^{(i)}$ are the random variables of the hypotheses.

Claim: The random variables

$$\sum_{z \in \mathbb{P}(\mathbb{R}^{m+1}) :\, F(z)=0} \#\Lambda^{-1}(z) \quad \text{and} \quad \#f^{-1}(0)$$

coincides almost every where $\omega \in \Omega$:
Let $V_c \subset \mathbb{P}(\mathbb{R}^{m+1})$ be the set of critical values of $\Lambda : U \to \mathbb{P}(\mathbb{R}^{m+1})$. By Sard's lemma, V_c has measure zero in $\mathbb{P}(\mathbb{R}^{m+1})$. Then outside the event

$$\{\omega \in \Omega :\ F^{-1}(0) \cap V_c \neq \emptyset\},$$

we have that $\sum_{z \in \mathbb{P}(\mathbb{R}^{m+1}) :\, F(z)=0} \#\Lambda^{-1}(z)$ is finite and equal to $\#f^{-1}(0)$. Therefore, it is enough to prove that the probability of the event $\{\omega \in \Omega :\ F^{-1}(0) \cap V_c \neq \emptyset\}$ is zero. Taking the push-forward measure on $\mathcal{H}_{(d)}$, the space of homogeneous polynomial systems where F lives, it is enough to prove that the measure of the set $A = \{h \in \mathcal{H}_{(d)} :\ h^{-1}(0) \cap V_c \neq \emptyset\}$ is zero. By the reproducing kernel property of (real) Weyl inner product (see (3.1.4) for the complex analogue), we have that the set of problematic systems is given by

$$A = \bigcup_{z \in V_c} K_z^{\perp},$$

where $K_z \in \mathcal{H}_{(d)}$ is the system of polynomials $K_z(x) = (\langle x, z\rangle^{d_i})_i$, $(i = 1, \ldots, m)$, and $K_z^{\perp} = \{h \in \mathcal{H}_{(d)} :\ \langle h_i, (K_z)_i \rangle_W = 0\}$. Note that $K_z^{\perp}$ is codimension m

subspace of $\mathcal{H}_{(d)}$, and the hausdorff dimension of $\{K_z : z \in V_c\}$ is less than m. Therefore $A \subset \mathcal{H}_{(d)}$ is union of codimension m subspaces parameterized on a set with Haudsorff dimension less than m. Since the map $z \mapsto K_z$ is differentiable, then we can conclude that A has measure zero on $\mathcal{H}_{(d)}$ proving the claim.

Then, using Rice formula for weighted roots (4.2.3), and *Claim II* and *Claim III* of the proof of *Theorem 11* we get

$$\begin{aligned}
\mathbb{E}\Big(\sum_{z\in\mathbb{P}(\mathbb{R}^{m+1}):\,F(z)=0} \#\Lambda^{-1}(z)\Big) &= \frac{1}{2}\mathbb{E}\Big(\sum_{z\in S^m:\,F(z)=0} \#\Lambda^{-1}([z])\Big)\\
&= \frac{1}{2}\int_{z\in S^m} \mathbb{E}\left(\#\Lambda^{-1}([z])\,|\det(DF(z)|_{z^\perp})|\,\middle|\,F(z)=0\right) p_{F(z)}(0)\,dz\\
&= \frac{1}{2}\int_{z\in S^m} \#\Lambda^{-1}([z])\,\mathbb{E}\left(|\det(DF(z)|_{z^\perp})|\,\middle|\,F(z)=0\right) p_{F(z)}(0)\,dz\\
&= \frac{1}{2}\mathbb{E}\left(|\det(DF(e_0)|_{e_0{}^\perp})|\right) p_{F(e_0)}(0)\int_{z\in S^m} \#\Lambda^{-1}([z])\,dz\\
&= \frac{1}{2}\frac{\mathbb{E}(N^F(S^m))}{\operatorname{vol}(S^m)}\int_{z\in S^m} \#\Lambda^{-1}(z)\,dz = \frac{\sqrt{d_1\cdots d_m}}{\operatorname{vol}(\mathbb{P}(\mathbb{R}^{m+1}))}\int_{z\in\mathbb{P}(\mathbb{R}^{m+1})} \#\,\Lambda^{-1}(\{z\})\,dz.
\end{aligned}$$

Therefore, the proof follows from *Theorem 11* and the *Claim*. □

Some Examples

Bernstein Polynomials:

Let us consider the set of real polynomial systems in m variables,

$$f_i(x_1,\ldots,x_m) = \sum_{\|j\|\le d_i} a_j^{(i)} x_m^{j_m}(1-x_1-\ldots-x_m)^{d-\|j\|} \qquad (i=1,\ldots,m).$$

Take the coefficients $a_j^{(i)}$ to be independent, Gaussian random variables with variance $\binom{d_i}{j}$.

Then, *Theorem 13* follows from *Theorem 15* taking, for $x \in \mathbb{R}^m$, $\varphi_i(x) = x_i$ for $i = 1\ldots,m$ and $\varphi_0(x) = 1 - x_1 - \ldots - x_m$.

Non-Polynomial Examples

- Consider the random polynomial

  $$f(t) = \sum_{j=0}^{d} a_j \cos(t)^{d-j} \sin(t)^j,$$

  where a_j are independent, centered, Gaussian random variables with variance $\binom{d}{j}$, $(j = 0, \dots, d)$.

  Then, considering $\varphi_0(t) = \cos(t)$ and $\varphi_1(t) = \sin(t)$, $t \in [0, \pi]$, we get from *Theorem 15* that

  $$\mathbb{E}(N^f([0, \pi])) = \sqrt{d}.$$

- Consider the random polynomial

  $$f(t) = \sum_{j=0}^{d} a_j t^{d-j} e^{jt},$$

  where $\{a_j\}$ are independent, centered, Gaussian random variables with variance $\binom{d}{j}$.

  Then, taking $\varphi_0(t) = t$, $\varphi_1(t) = e^t$, for $t \in \mathbb{R}$, we conclude from *Theorem 15* that

  $$\mathbb{E}(N^f(\mathbb{R})) = \sqrt{d}(1 - \mathrm{Arctan}(e)/\pi).$$

4.7 Random Real Algebraic Varietes

Let us assume now that we have less equations than variables, that is, let $f : \mathbb{R}^n \to \mathbb{R}^k$ be a random system of polynomials such that $k < n$. In this case $\mathcal{Z}(f_1, \dots, f_k) = f^{-1}(0)$ is a random algebraic variety of positive dimension. A natural questions come into account:

What is the average volume of $\mathcal{Z}$?

In the next lines we attack this problem by means of the Rice Formulas. In Bürgisser [2006] and Bürgisser [2007] one can find a nice study of this an other important questions concerning geometric properties of random algebraic varieties.

We will restrict ourselves to the particular case of the Shub-Smale distribution.

Let us consider the random system of k homogeneous polynomial equations in $m+1$ unknowns $f : \mathbb{R}^{m+1} \to \mathbb{R}^k$, given by

$$f_i(x) := \sum_{\|j\|=d_i} a_j^{(i)} x^j, \qquad (i = 1, \ldots, k). \tag{4.7.1}$$

Assume that this system has the Shub-Smale distribution, that is, $\{a_j^{(i)}\}$ are Gaussian, centered, independent random variables having variances

$$\mathbb{E}\left[(a_j^{(i)})^2\right] = \binom{d_i}{j} = \frac{d_i!}{j_0! \cdots j_m!}.$$

Since f is homogeneous, we can restrict to the sphere $S^m \subset \mathbb{R}^{m+1}$ our study of the random set $\mathcal{Z}(f_1, \ldots, f_k)$. Note that, generically, $\mathcal{Z}(f_1, \ldots, f_k) \cap S^m$ is a smooth manifold of dimension $m-k$. Let us denote by λ_{m-k} the $m-k$ geometric measure.

Theorem 16. *Let $f : \mathbb{R}^{m+1} \to \mathbb{R}^k$ be the system (4.7.1) with the Shub-Smale distribution. Then, one has*

$$\mathbb{E}(\lambda_{m-k}(\mathcal{Z}(f_1, \ldots, f_k) \cap S^m)) = \sqrt{d_1 \cdots d_k}\, vol(S^{m-k+1}).$$

This result was first observed by Kostlan [1993] in the particular case $d_1 = \ldots = d_k$. We give a proof of this proposition based on the Rice formula for the geometric measure. We will see that the proof is almost the same as the proof of Shub-Smale *Theorem 11.* (See *Remark 4.4.1*). The difference lies in the fact that we should compute the expected value of the determinant of a different random matrix. At the end of this section we will see how one can obtain another proof of this theorem from *Theorem 11* and the fairly known Crofton-Poincare formula of integral geometry.

Proof of Theorem 16. Using the Rice formula for the geometric measure (4.2.4) we get:

$$\mathbb{E}(\lambda_{m-k}(\mathcal{Z}(f_1,\ldots,f_k)\cap S^m)) =$$
$$= \int_{S^m} \mathbb{E}\left(|\det[(Df(x)|_{x^\perp}\cdot(Df(x)|_{x^\perp})^T|^{1/2}] \,\middle|\, f(x)=0\right) p_{f(x)}(0)\, dS^m(x).$$

From *Claim I* of the proof of *Theorem 11* we get that the law of the process in invariant under the action of the orthogonal group in $\mathbb{R}^{m+1}$. From *Claim II* of the proof of the same theorem we get that the law of the derivative $Df(x)|_{x^\perp}$ (restricted to orthogonal complement of $x\in S^m$) is independent of the law of the condition $f(x)$. Then, we have

$$\mathbb{E}(\lambda_{m-k}(\mathcal{Z}(f_1,\ldots,f_k)\cap S^m)) =$$
$$= \mathrm{vol}(S^m)\,\mathbb{E}(|\det[(Df(e_0)|_{e_0^\perp})\cdot(Df(e_0)|_{e_0^\perp})^T]|^{1/2})p_{f(e_0)}(0).$$

Moreover, since the random vector $f(e_0)\in\mathbb{R}^k$ has standard normal distribution, we get

$$\mathbb{E}(\lambda_{m-k}(\mathcal{Z}(f_1,\ldots,f_k)\cap S^m)) =$$
$$= \frac{vol(S^m)}{\sqrt{2\pi}^k}\,\mathbb{E}(|\det[(Df(e_0)|_{e_0^\perp})\cdot(Df(e_0)|_{e_0^\perp})^T]|^{1/2}).$$

Hence, we reduce the computations to a problem in random matrices. From (4.4.4) we obtain that

$$\mathbb{E}(|\det[(Df(e_0)|_{e_0^\perp})\cdot(Df(e_0)|_{e_0^\perp})^T]|^{1/2}) = \sqrt{d_1\cdots d_k}\,\mathbb{E}(\det(G_{k\times m}\cdot {G_{k\times m}}^T)^{1/2}),$$

where $G_{k\times m}$ is the $k\times m$ random matrix whose coefficients are i.i.d standard Gaussian random variables.

By standard conditioning arguments one can prove that

$$\mathbb{E}(\det(G_{k\times m}\cdot {G_{k\times m}}^T)^{1/2}) = \prod_{i=m-k+1}^{m}\mathbb{E}(\|\xi_j\|),$$

where $\|\xi_j\|$ is the Euclidean norm of a standard Gaussian random vector in $\mathbb{R}^j$.

It is easy to see that $\mathbb{E}(\|\xi_j\|) = \sqrt{2}\Gamma\big((i+1)/2\big)/\Gamma\big(i/2\big)$. Then, we conclude

$$\begin{aligned}\mathbb{E}(\lambda_{m-k}(\mathcal{Z}(f_1,\dots,f_k)\cap S^m)) &= \sqrt{d_1\cdots d_k}\frac{vol(S^m)}{(2\pi)^{k/2}}2^{k/2}\frac{\Gamma\left(\frac{m+1}{2}\right)}{\Gamma\left(\frac{m-k+1}{2}\right)}\\ &= \sqrt{d_1\cdots d_k}\mathrm{vol}(S^{m-k+1}),\end{aligned}$$

proving the result. □

Recall the Crofton-Poincare Formula:

Theorem 17 (Crofton-Poincare Formula). *Let Z and P be compact smooth submanifolds of S^m of dimension q and p, such that $q+p \geq m$. Then*

$$\int_{g\in O(n+1)} \lambda_{p+q-m}(Z\cap gP)\,dg = \frac{vol(S^{q+p-m})}{vol(S^q)\,vol(S^p)}\,\lambda_q(Z)\,\lambda_p(P),$$

where $O(n+1)$ is the orthogonal group in $\mathbb{R}^{m+1}$.

This is a classical formula in integral geometry. For references see for example Howard [1993].

Alternative Proof of Theorem 16. Let $\{e_0,\dots,e_m\}$ be the canonical basis of $\mathbb{R}^{m+1}$, and denote by S^k the k-dimensional sphere on S^m given by intersecting S^m with the orthogonal complementary subspace spanned by $\{e_k,\dots,e_m\}$.

Taking $Z=\mathcal{Z}(f_1,\dots,f_k)\cap S^m$, $P=S^k$, $q=m-k$, $p=k$ we get that for almost every $\omega\in\Omega$,

$$\int_{g\in O(n+1)} \#(\mathcal{Z}(f_1,\dots,f_k)\cap gS^k)\,dg = 2\frac{\lambda_{m-k}(\mathcal{Z}(f_1,\dots,f_k)\cap gS^m)}{\mathrm{vol}(S^{m-k})}. \qquad (4.7.2)$$

Let us compute $\mathbb{E}(\#(\mathcal{Z}(f_1,\dots,f_k)\cap gS^k))$ for $g\in O(n+1)$.

Since the law of $f_1,\dots,f_k$ is invariant under the action of the orthogonal group, we have that $\mathbb{E}(\#(\mathcal{Z}(f_1,\dots,f_k)\cap gS^k)) = \mathbb{E}(\#((f\circ g^{-1})^{-1}(0)\cap gS^k))$, and therefore we conclude that

$$\mathbb{E}(\#(\mathcal{Z}(f_1,\dots,f_k)\cap gS^k)) = \mathbb{E}(\#(\mathcal{Z}(f_1,\dots,f_k)\cap S^k)) \quad \text{for all} \quad g\in O(n+1).$$

Therefore, if one pick a circle at random, independently of the law of the random field $f = (f_1, \dots, f_k)$, we obtain that the expected value of the number of intersection points of $\mathcal{Z}(f_1, \dots, f_k)$ with the random circles is equal to $\mathbb{E}(\#(\mathcal{Z}(f_1, \dots, f_k) \cap S^k))$.

Let us randomize the circles in a way such that we now that answer. Let us consider the system of random equations $F : \mathbb{R}^{m+1} \to \mathbb{R}^m$, given by

$$\begin{cases} f_1(x) = 0 \\ \vdots \\ f_k(x) = 0 \\ \langle x, \eta_k \rangle = 0 \\ \vdots \\ \langle x, \eta_m \rangle = 0 \end{cases}$$

where $\eta_k, \dots, \eta_m$ are i.i.d standard gaussian vectors in $\mathbb{R}^{m+1}$, independent of the coefficients $\{a_j^{(i)}\}_{i=1,\dots,k; \|j\|=d_i}$ of f. Then it is immediate to check that the random field F has the (homogeneous) Shub-Smale distribution with degrees $d_1, \dots, d_k$ and $m - k$ degrees equal 1. Then, from *Theorem 11* we get that $\mathbb{E}(N^F(S^m))$ is equal to 2 times the square root of the product of the degrees, that is, $\mathbb{E}(N^F(S^m)) = 2\sqrt{d_1 \cdots d_k}$. Hence we conclude that

$$\mathbb{E}(\#(\mathcal{Z}(f_1, \dots, f_k) \cap gS^k)) = 2\sqrt{d_1 \cdots d_k}, \tag{4.7.3}$$

for all $g \in O(n+1)$.

Then from (4.7.2) and (4.7.3) we get

$$\mathbb{E}(\lambda_{m-k}(\mathcal{Z}(f_1, \dots, f_k) \cap S^m)) = \sqrt{d_1 \cdots d_k}\, \mathrm{vol}(S^{m-k}).$$

$\square$

Chapter 5

Complex Random Systems of Polynomials

In this chapter we study complex random systems of polynomial equations. The main objective is to introduce the technics of Rice Formulas in the realm of complex random fields. At the end we give a probabilistic approach of Bézout's theorem using Rice Formulas.

5.1 Introduction and Preliminaries

Let $\mathcal{H}_{(d)}$ be the space of m homogeneous polynomials of degrees $(d) = (d_1, \dots, d_m)$ in $(m+1)$ complex variables. Let us denote as usual

$$f_\ell(z) = \sum_{\|j\|=d_\ell} a_j^{(\ell)} z^j, \quad \ell = 1, \dots, m, \tag{5.1.1}$$

where d_ℓ is the degree of the polynomial f_ℓ, $j = (j_0. \dots, j_m) \in \mathbb{N}^{m+1}$ is a multi-index of nonnegative integers, $z = (z_0. \dots, z_m) \in \mathbb{C}^{m+1}$ is a point in $\mathbb{C}^{m+1}$, $a_j^{(\ell)} = a_{j_0 \dots j_m}^{(\ell)} \in \mathbb{C}$, and $\|j\| = \sum_{k=0}^m j_k$, $z^j = z_0^{j_0} \dots z_m^{j_m}$.

If one randomize the coefficients $\{a_j^{(\ell)}\}$ on the complex plane, we obtain a complex polynomial random field. In the next lines we introduce the basic notions of random variables in the complex plane. After that we analyze a particular complex polynomial random field, rewriting Rice formulas for this context.

5.1.1 Gaussian Complex Random Variables

We say that the complex random variable $Z = X + iY$ has distribution $\mathcal{N}_{\mathbb{C}}(0, \sigma^2)$ when the real part X and the imaginary part Y are i.i.d. Gaussian centered random variables with variance $\sigma^2/2$.

Thus, if $Z \sim \mathcal{N}_{\mathbb{C}}(0, \sigma^2)$ then the density with respect to the Lebesgue measure on the complex plane is

$$p_Z(z) = \frac{1}{\pi} e^{-|z|^2/\sigma^2}, \qquad z \in \mathbb{C}. \tag{5.1.2}$$

It is easy to check that in this case $\mathbb{E}Z = \mathbb{E}X + i\mathbb{E}Y = 0$, and

$$\mathbb{E}\left[Z\overline{Z}\right] = \mathbb{E}(|Z|^2) = \sigma^2, \quad \mathbb{E}\left[ZZ\right] = 0. \tag{5.1.3}$$

In general, we say that $Z = X + iY$ is a *complex Gaussian random variable* if the pair (X, Y) is Gaussian random vector on $\mathbb{R}^2$.

The next lemma is a useful condition to verify that that two complex Gaussian random variables are independent.

Lemma 5.1.1. *Let Z and Z' be two complex centered Gaussian random variables. Then Z and Z' are independent if and only if*

$$\begin{cases} \mathbb{E}(Z\overline{Z'}) = 0, \\ \mathbb{E}(ZZ') = 0 \end{cases}$$

Proof. Let us write $Z = X + iY$ and $Z' = X' + iY'$. Note that

$$\begin{cases} Z\overline{Z'} = XX' + YY' + i(YX' - XY'); \\ ZZ' = XX' - YY' + i(XY' + YX'). \end{cases}$$

If Z is independent of Z' then it is clear that $\mathbb{E}(Z\overline{Z'}) = \mathbb{E}(ZZ') = 0$. On the other hand, if $\mathbb{E}(Z\overline{Z'}) = \mathbb{E}(ZZ') = 0$ then taking expected value in last expressions we get that $\mathbb{E}(XX') = \mathbb{E}(XY') = \mathbb{E}(YX') = \mathbb{E}(YY') = 0$ proving the independence od Z and Z'. $\square$

5.1.2 Real and Hermitian Structures

The space $\mathbb{C}^{m+1}$ is identified with $\mathbb{R}^{2m+2}$ by

$$(z_0, \ldots, z_m) \in \mathbb{C}^{m+1} \mapsto \hat{z} = (x_0, \ldots, x_m, y_0, \ldots, y_m) \in \mathbb{R}^{2m+2},$$

where we have denoted $z_\ell = x_\ell + iy_\ell$, for $0 \leq \ell \leq m$.

It is easy to see that the real part of the Hermitian inner product $\langle \cdot, \cdot \rangle$ in $\mathbb{C}^{m+1}$ is the canonical inner product on $\mathbb{R}^{2m+2}$, that is,

$$\mathrm{Re}\langle z, w\rangle_{\mathbb{C}^{m+1}} = \langle \hat{z}, \hat{w}\rangle_{\mathbb{R}^{2m+2}}. \tag{5.1.4}$$

In what follows we will suppress the subindex and write $\langle \cdot, \cdot \rangle$, and we will use the same symbol z for represent a vector in $\mathbb{C}^{m+1}$ and $\mathbb{R}^{2m+2}$. It should be understood from the context.

Remark 5.1.1. Let $\mathcal{U}(\mathbb{C}^{m+1})$ be the unitary group of $\mathbb{C}^{m+1}$. From (5.1.4) $\mathcal{U}(\mathbb{C}^{m+1})$ acts on $\mathbb{R}^{2m+2}$ by isometries of the canonical real inner product on $\mathbb{R}^{2m+2}$. Moreover, that action is transitive on the sphere $S^{2m+1} \subset \mathbb{R}^{2m+2}$.

5.1.3 Weyl Distribution

We say that the system of polynomials $f = (f_1, \ldots, f_m) : \mathbb{C}^{m+1} \to \mathbb{C}^m$ given in (5.1.1) has the *Weyl distribution* if the coefficients $a_j^{(\ell)}$ are independent and $a_j^{(\ell)} \sim \mathcal{N}_{\mathbb{C}}\left(0, \binom{d_\ell}{j}\right)$. That is, the coefficients $a_j^{(\ell)}$ are independent centered Gaussian complex variables such that

$$\mathbb{E}a_j^{(\ell)}\overline{a_j^{(\ell)}} = \binom{d_\ell}{j}, \qquad \mathbb{E}(a_j^{(\ell)})^2 = 0. \tag{5.1.5}$$

Lemma 5.1.2. *Let $f = (f_1, \ldots, f_m)$ with the Weyl distribution, then for all $z, w \in \mathbb{C}^{m+1}$ we have*

$$\mathbb{E}f_k(z)\overline{f_k(w)} = \langle z, w\rangle^{d_k},$$
$$\mathbb{E}f_k(z)f_k(w) = 0,$$

for $k = 1, \dots, m$.

Proof. We omit the index k for notational convenience.

$$\begin{aligned}\mathbb{E}\big(f(z)\overline{f(w)}\big) &= \mathbb{E}\left[\sum_{\|j\|=d} a_j z^j \cdot \overline{\sum_{\|j'\|=d} a_{j'} w^{j'}}\right] \\ &= \sum_{\|j\|=d}\sum_{\|j'\|=d} \mathbb{E}(a_j\overline{a_{j'}}) z^j \overline{w^{j'}}.\end{aligned}$$

Since the a_j are independent and centered, for $j \neq j'$ we have $\mathbb{E}(a_j\overline{a_{j'}}) = \mathbb{E}(a_j)\mathbb{E}(\overline{a_{j'}}) = 0$. If $j = j'$, from (5.1.3) we have $\mathbb{E}(|a_j|^2) = \binom{d}{j}$, hence

$$\begin{aligned}\mathbb{E}\big(f(z)\overline{f(w)}\big) &= \sum_{\|j\|=d} \mathbb{E}(|a_j|^2) z^j w^j = \sum_{\|j\|=d} \binom{d}{j} z^j w^j \\ &= \langle z, w\rangle^d_{\mathbb{C}^{m+1}}\,.\end{aligned}$$

For the second assertion of this lemma note that

$$\mathbb{E} f(z) f(w) = \sum_{\|j\|=d} \mathbb{E}(a_j^2) z^j w^j.$$

Then, the second assertion follows from (5.1.3). $\square$

Remark 5.1.2. From *Lemma 5.1.2* we conclude that the random field f with the Weyl distribution is invariant under the action of the unitary group in $\mathbb{C}^{m+1}$, and therefore the associated real random field is invariant under the action of a sub group of the orthogonal group in $\mathbb{R}^{2m+2}$ that acts transitively on the sphere S^{2m+1} (see *Remark 5.1.1*).

Density $f(z)$

Let $z \in \mathbb{C}^{m+1}$, such that $\|z\| = 1$. By *Lemma 5.1.2* the complex random variable $f_1(z), \dots, f_m(z)$ are i.i.d complex standard Gaussian. Then from (5.1.2) we obtain

$$p_{f(z)}(0) = \frac{1}{\pi^m}. \tag{5.1.6}$$

5.1.4 Real and Complex Derivatives of Holomorphic Maps

Let $f \in \mathcal{H}_{(d)}$. Then f is a holomorphic map in several variables. If $z \in \mathbb{C}^{m+1}$ is a zero of f, then we can restrict the complex derivative $f'(z) : \mathbb{C}^{m+1} \to \mathbb{C}^m$ to $f'(z)|_{z^\perp} : z^\perp \to \mathbb{C}^m$, where $z^\perp$ is the Hermitian complement of z in $\mathbb{C}^{m+1}$. Therefore, we can define the complex determinant

$$\det(f'(z)|_{z^\perp}),$$

as the determinant of the associated $m \times m$ complex matrix.

Also f is a map from $\mathbb{R}^{2m+2}$ onto $\mathbb{R}^{2m}$ real differentiable. If $z \in \mathbb{R}^{2m+2}$ is a root of f, then also iz is a root ($i = \sqrt{-1}$), and z and iz are real independent. Therefore, f vanishes on a real subspace of dimension 2, namely the real space associated to the complex linear subspace generated by z. In this way we can restrict the real derivative $Df(z)$ to $z^\perp$ and obtain in this way a map $Df(z)|_{z^\perp} : z^\perp \to \mathbb{R}^{2m}$. Fixed a canonical basis on those spaces, let $\det(Df(z)|_{z^\perp})$ be its determinant. Then

Lemma 5.1.3.

$$\det(Df(z)|_{z^\perp}) = |\det(f'(z)|_{z^\perp})|^2.$$

This result is fairly known, in complex analysis, see for example Range [1986].

5.2 Rice Formulas for Complex Random Polynomial Fields

Let $f \in \mathcal{H}_{(d)}$.

Since f is homogeneous, if $z \in \mathbb{C}^{m+1}$ is a root of the system f, then $f(\lambda z) = 0$ for all $\lambda \in \mathbb{C}$. Hence, the roots of f are complex lines in $\mathbb{C}^{m+1}$ through the origin.

This suggest to work on the projective space $\mathbb{P}(\mathbb{C}^{m+1})$ or on the sphere S^{2m+1}. From now on, we will work on the sphere.

Note that, if $z \in S^{2n+1}$ then $\lambda z \in S^{2n+1}$ for all λ such that $|\lambda| = 1$, hence the zeros of f in S^{2m+1} are, generically, a union of real circles, namely,

$$\bigcup_{z\in\mathbb{P}(\mathbb{C}^{m+1}):\, f(z)=0} \{e^{i\theta}z \,:\, \theta\in[0,2\pi)\},$$

where the union is indexed in projective roots.

Generically, these are 1-dimensional real circles embedded on S^{2m+1}, with Riemannian length 2π.

Now, assume that the system f has the Weyl distribution (5.1.5). Since, almost surely, the intersection of these circles have zero Lebesgue measure (see Blum *et al.* [1998]), then the number of projective complex zeros of f is equal almost surely to $1/(2\pi)$ times the geometric 1-one dimensional measure of $f^{-1}(0)\cap S^{2m+1}$.

Denoting by N the number of zeros of f and λ_1 the geometric 1-dimensional measure, we have from Rice formula for the geometric measure (4.2.4) that

$$\begin{aligned}2\pi\mathbb{E}(N) &= \mathbb{E}(\lambda_1(f^{-1}(0)\cap S^{2m+1}))\\ &= \int_{S^{2m+1}} \mathbb{E}\left[|\det(Df(z)\cdot Df(z)^T)|^{1/2}\big|f(z)=0\right] p_{f(z)}(0)dS^{2m+1}.\end{aligned}$$

Here Df stands for the (real) derivative of f along the manifold S^{2m+1} and dS^{2m+1} for the geometric measure on S^{2m+1}.

Then from *Lemma 5.1.3* we obtain:

Proposition 5.2.1. *Let f be the homogeneous system polynomials (5.1.1) with the Weyl distribution (5.1.5). Then*

$$\mathbb{E}(N) = \frac{1}{2\pi}\int_{S^{2m+1}} \mathbb{E}\left[|\det(f'(z)|_{z^\perp})|^2\big|f(z)=0\right] p_{f(z)}(0)\, dS^{2m+1}(z). \tag{5.2.1}$$

5.3 A Probabilistic Approach to Bézout's Theorem.

This section follows closely a joint work under construction with Federico Dalmao and Mario Wschebor [Armentano *et al.*, 2012]. The main objective of this work is to give a probabilistic proof of Bézout's theorem. More precisely:

Theorem 18 (Bézout Probabilistic)**.** *Assume that f has the Weyl distribution and denote by N the number of projective zeros of f, then*

$$N = \mathcal{D} \text{ a.s.}$$

where $\mathcal{D} = \prod_{\ell=1}^{m} d_i$ is Bézout number.

The proof we have attempted was divided into two steps:
- First prove that the expected value of N is $\mathcal{D}$;
- Secondly, prove that the variance of the random variable $N - \mathcal{D}$ is zero.

Both steps can be analyzed with Rice formulas. The first step follows similarly to the proof of *Theorem 11.* For the second step we use a version of the Rice formula for the k-moment given in (4.2.2).

The second step involves many computations. Even though we could not finish the proof of the second step, we will show how to proceed in the computations and we will show the main difficulties. On the particular case of $m = 1$, that is, the Fundamental Theorem of Algebra, we finish the proof.

5.3.1 Expected Number of Projective Zeros

Proposition 5.3.1. *Assume that f has the Weyl distribution and denote by N the number of projective zeros of f, then*

$$\mathbb{E}(N) = \mathcal{D}.$$

Proof. Denoting by N the number of zeros of f and λ_1 the geometric 1-dimensional measure, we have from (5.2.1) that

$$\mathbb{E}(N) = \frac{1}{2\pi} \int_{S^{2m+1}} \mathbb{E}\left[|\det(f'(z)|_{z^\perp})|^2 \middle| f(z) = 0 \right] p_{f(z)}(0)\, dS^{2m+1}(z).$$

Note that from *Lemma 5.1.2*, $\mathbb{E}(f_k(z)\overline{f_k(w)}) = \langle z, w\rangle^{d_k}$. Therefore $\mathbb{E}(|f_k(z)|^2) = \|z\|^{2d_k}$. Hence, differentiating under the sign of expectation, in the a direction orthogonal to $z^\perp$, we get that $\mathbb{E}(\partial_\ell f_k(z)\overline{f_k(z)}) = 0$, where ∂_ℓ denotes some derivative along the direction $z^\perp$. Moreover, $\mathbb{E}(\partial_\ell f_k(z)\, f_k(z))$ is trivially zero since $\mathbb{E}(f_k(z) f_k(z)) = 0$.

Thereby, from *Lemma 5.1.1*, we conclude that the linear map $f'(z)|_{z^\perp}$ is independent of $f(z)$. The conditional expectation $\mathbb{E}\left[|\det(f'(z)|_{z^\perp})|^2 \big| f(z)=0\right]$ is equal to the (inconditional) expectation $\mathbb{E}[|\det(f'(z)|_{z^\perp})|^2]$.

Moreover, since $\mathcal{U}(\mathbb{C}^{m+1})$ acts transitively on S^{2m+1}, and the random field f is invariant under the action of this group we conclude from (5.1.6) and *Lemma 5.1.3* that

$$\mathbb{E}N = \frac{1}{2\pi}\frac{\mathrm{vol}(S^{2m+1})}{\pi^m}\mathbb{E}\left[|\det(f'(e_0)|_{e_0^\perp})|^2\right].$$

Let us write the derivative $f'(e_0)|_{e_0^\perp}$ on the basis $\{e_1,\dots,e_m\}$ of $e_0^\perp$. Similar to the computations we did in the real case in (4.4.4), we get

$$\mathbb{E}\left(\frac{\partial f_k}{\partial z_\ell}(e_0)\overline{\frac{\partial f_k}{\partial z_{\ell'}}(e_0)}\right) = \frac{\partial}{\partial z_\ell}\overline{\frac{\partial}{\partial z_{\ell'}}}\langle z,w\rangle^{d_k}\big|_{z=w=e_0} = d_k\delta_{\ell,\ell'},$$

for $k,\ell,\ell'=1\dots,m$. Then, expressing $f'(e_0)|_{e_0^\perp}$ in the canonical basis $\{e_1\dots,e_m\}$, it follows that

$$f'(e_0)|_{e_0^\perp} = \Delta(\sqrt{d_i})G_m,$$

where G_m an $m\times m$ matrix which entries are i.i.d. complex standard Gaussian, and hence

$$\mathbb{E}(N) = \mathcal{D}\frac{1}{2\pi}\frac{\mathrm{vol}(S^{2m+1})}{\pi^m}\mathbb{E}|\det(G_m)|^2.$$

Note that $|\det(G_m)|^2 = \det(G_m)\det(\overline{G_m})$, then,

$$\begin{aligned}\mathbb{E}|\det(G_m)|^2 &= \mathbb{E}\det(G_m)\det(\overline{G_m})\\ &= \sum_{\pi,\pi'\in S_m}(-1)^\pi(-1)^{\pi'}\mathbb{E}(g_{1\pi(1)}\dots g_{m\pi(m)}\overline{g_{1\pi'(1)}}\dots\overline{g_{m\pi'(m)}})\\ &= \sum_{\pi\in S_m}\mathbb{E}|g_{1\pi(1)}|^2\dots|g_{m\pi(m)}|^2 = \sum_{\pi\in S_m}1 = m!.\end{aligned}$$

The third equality follows from the independence of the coefficients of G_m and the fact that they are centered.

Then we conclude

$$\mathbb{E}N = \mathcal{D}\frac{\mathrm{vol}(S^{2m+1})m!}{2\pi^{m+1}} = \mathcal{D}.$$

□

Remark 5.3.1. Recall that in the proof of *Theorem 11* we reduce the problem of computing the average number of roots, to a problem of random matrices, namely, compute $\mathbb{E}(|\det G|)$ where G is a $m \times m$ real Gaussian standard matrix. However, in the complex case, we reduce the problem of computing the average number of roots to the computation $\mathbb{E}(|\det G|^2)$. This case is much simpler as compared with the real case since in this case one can develop the terms inside the determinant and interchange the sum with the expectations sign. This is what we did.

5.3.2 Second Moment Computations

We compute now $\mathbb{E}N^2$, with N the number of projective roots of the system f.

For this computations we need a Rice formula for the second moment adapted to this case. For short we write $f'(z)$ to the restriction $f'(z)|_{z^\perp}$.

Lemma 5.3.1. *One has*

$$4\pi^2(\mathbb{E}(N^2) - \mathcal{D}) = \int_{S^{2m+1}\times S^{2m+1}} \mathbb{E}\left[|\det(f'(z))|^2|\det(f'(w))|^2 \mid f(z) = f(w) = 0\right] p_{f(z),f(w)}(0,0)dzdw,$$

Proof. Following Azaïs & Wschebor [2009], let $F : S^{2m+1} \times S^{2m+1} \to \mathbb{R}^{2m}$ be the map given by $F(z,w) = (f(z), f(w))$ and let $\Delta_\delta \subset S^{2m+1} \times S^{2m+1}$ be the set defined by $\Delta_\delta = \{(s,t) \in S^{2m+1} \times S^{2m+1} : \|s-t\| > \delta\}$. Then, applying Rice Formula for the geometric measure of $F^{-1}(0,0)$ (4.2.4) we get:

$$\begin{aligned}\mathbb{E}\lambda_2(F^{-1}(0,0)\cap\Delta_\delta) &= \\ &= \int_{\Delta_\delta} \mathbb{E}\left[|\det(f'(z))|^2|\det(f'(w))|^2 \mid f(z) = f(w) = 0\right] \\ &\quad p_{f(z),f(w)}(0,0)dzdw,\end{aligned}$$

Taking limit $\delta \downarrow 0$ we observe that

$$\mathbb{E}(\lambda_1(F^{-1}(0,0)\cap\Delta_\delta)) \uparrow \mathbb{E}(\lambda_2(F^{-1}(0,0))) - \mathbb{E}(\lambda_2(F^{-1}(0,0)\cap\Delta)),$$

where $\Delta = S^{2m+1} \times S^{2m+1} - \Delta_0$ is the diagonal set. Hence

$$\lim_{\delta \downarrow 0} \mathbb{E}(\lambda_1(F^{-1}(0,0) \cap \Delta_\delta)) = 4\pi^2 \mathbb{E}(N^2) - 4\pi^2 \mathfrak{D}$$

Moreover, since Δ has zero Lebesgue measure on $S^{2m+1} \times S^{2m+1}$ we conclude the lemma. $\square$

Joint Density $p_{f(z),f(w)}$

In the next lemma we compute the joint density of the pair $(f(z), f(w))$ at $(0,0)$.

Lemma 5.3.2. *The density $p_{f(z),f(w)}$ at $(0,0)$ is given by*

$$p_{f(z),f(w)}(0,0) = \frac{1}{\pi^{2m}} \prod_{\ell=1}^{m} \frac{1}{1 - |\langle z, w\rangle|^{2d_\ell}}.$$

Proof. Since different rows of the system are independent we have

$$p_{f(z),f(w)}(0,0) = \prod_{\ell=1}^{m} p_{f_\ell(z),f_\ell(w)}(0,0).$$

Furthermore, the covariance matrix of $(f_\ell(z), f_\ell(w))$ is $\Sigma = \begin{bmatrix} 1 & \langle z, w\rangle^{d_\ell} \\ \langle w, z\rangle^{d_\ell} & 1 \end{bmatrix}$. Therefore $p_{f_\ell(z),f_\ell(w)}(0,0) = \frac{1}{\pi^2(1-|\langle z,w\rangle|^{2d_\ell})}$. Hence

$$p_{f(z),f(w)}(0,0) = \frac{1}{\pi^{2m}} \prod_{\ell=1}^{m} \frac{1}{1 - |\langle z, w\rangle|^{2d_\ell}}.$$

$\square$

Conditional Expectation Computation

The natural procedure here is to perform the linear regression of $f'(z)$ and $f'(w)$ over $f(z)$ and $f(w)$. This procedure is quite standard in probability theory and statistics (see *Appendix B.2*). We leave this computations to the end of this chapter in the next section. One get that

$$\mathbb{E}\left[|\det(f'(z))|^2|\det(f'(w))|^2 \mid f(z)=f(w)=0\right] = \mathbb{E}\left[|\det(M(z))|^2|\det(M(w))|^2\right], \tag{5.3.1}$$

where $M(z)=(\zeta_{ij}^z)_{ij}, M(w)=(\zeta_{ij}^w)_{ij}$ are matrices with independent entries such that

$$\begin{aligned}\mathbb{E}\zeta_{ij}^z\overline{\zeta_{ij}^z}=\mathbb{E}\zeta_{ij}^w\overline{\zeta_{ij}^w} &= \begin{cases} d^2 & j\neq 1 \\ d_i^2\sigma_i^2 & j=1\end{cases}\\ \mathbb{E}\zeta_{ij}^z\overline{\zeta_{ij}^w} &= \begin{cases} d^2\langle z,w\rangle^{d_i} & j\neq 1 \\ d_i^2\tau_i & j=1\end{cases}\end{aligned}$$

where

$$\begin{aligned}\sigma_i^2 &= 1-\frac{d|\langle z,w\rangle|^{2d-2}}{1+|\langle z,w\rangle|^2+\cdots+|\langle z,w\rangle|^{2d_i-2}}\\ \tau_i &= \langle z,w\rangle^{d_i-2}\left[1-\frac{d}{1+|\langle z,w\rangle|^2+\cdots+|\langle z,w\rangle|^{2d_i-2}}\right]\end{aligned}$$

(Compare with [Azaïs & Wschebor, 2009, page 307]).

Case $m=1$

In this case we have that the conditional expectation is $\mathbb{E}(|\zeta|^2|\zeta'|^2)$, where ζ,ζ' are complex centered Gaussian random variables with variance σ^2 and covariance τ. Applying *Lemma 5.3.3* from the next section, for $S=\zeta/\sigma$, $T=\zeta'/\sigma$ we deduce that

$$\mathbb{E}(|\zeta|^2|\zeta'|^2)=\sigma^4+|\tau|^2.$$

Thus

$$\begin{aligned}4\pi^2(\mathbb{E}\mathcal{N}^2-\mathcal{D}) &= \int_{S^3\times S^3}\frac{\sigma^4+|\tau|^2}{\pi^2(1-|\langle z,w\rangle|^{2d})}\,dz\,dw\\ &= \frac{vol(S^3)}{\pi^2}\int_{S^3}\frac{\sigma^4+|\tau|^2}{(1-|\langle e_0,w\rangle|^{2d})}\,dw\end{aligned}$$

The integrand depend only on the modulus of the Hermitian inner product between e_0 and w, so we may apply the co-area formula for $\psi : S^3 \to \mathbb{D}$ such that $(e_0, w) \mapsto \langle e_0, w\rangle$. Denote $w = (x, y) \in \mathbb{C}^2$, then $x = \langle e_0, w\rangle$, then the Normal Jacobian is $\sqrt{1-|x|^2}$ (see [Blum *et al.*, 1998, Lemma 2, page 206]), and

$$\begin{aligned}
4\pi^2(\mathbb{E}\mathcal{N}^2 - \mathcal{D}) &= \frac{vol(S^3)}{\pi^2}\int_{x\in\mathbb{D}} \frac{\sigma^4+|\tau|^2}{(1-|x|^{2d})}\int_{\theta\in S(\sqrt{1-|x|^2})}\frac{1}{\sqrt{1-|x|^2}}\,d\theta\,dx\\
&= \frac{vol(S^3)vol(S^1)}{\pi^2}\int_{\mathbb{D}}\frac{\sigma^4+|\tau|^2}{(1-|x|^{2d})}dx.
\end{aligned}$$

Finally, changing to polar coordinates

$$4\pi^2(\mathbb{E}\mathcal{N}^2 - \mathcal{D}) = \frac{vol(S^3)(vol(S^1))^2}{\pi^2}\int_0^1 \rho\frac{\sigma^4+|\tau|^2}{(1-\rho^{2d})}dx.$$

One has $\int_0^1 \rho\frac{\sigma^4+|\tau|^2}{(1-\rho^{2d})}dx = \frac{1}{2}\cdot d(d-1)$, and therefore $\mathbb{E}N^2 = \mathcal{D}^2$ as claimed.

5.3.3 Auxiliary computations

Lemma 5.3.3. *Let (S, T) be centered, complex Gaussian random variables with variance 1 and covariance ρ. Denote S_r, T_r and S_{im}, T_{im} for the real and imaginary parts of S and T respectively, denote $\rho_{i,j} = \mathbb{E}(S_iT_j)$ for $i, j = r, im$. Then*

$$\begin{aligned}
\rho_{r,r} &= \rho_{im,im} = \frac{1}{2}\mathbb{R}e(\rho)\\
\rho_{r,im} &= -\rho_{im,r} = -\frac{1}{2}Im(\rho)
\end{aligned}$$

Lemma 5.3.4. *Let (S, T) be centered, Gaussian random variables with variance 1 and covariance ρ. Then*

1. *on the real case $\mathbb{E}(|S|^2|T|^2) = 1 + 2\rho^2$.*
2. *on the complex case $\mathbb{E}(|S|^2|T|^2) = 1 + |\rho|^2$.*

Proof. Real case
Let S, W be two real independent, centered, Gaussian random variables and write $T = \rho S + \sqrt{1-\rho^2}W$, then

$$\mathbb{E}(S^2T^2) = \rho^2\mathbb{E}S^4 + 2\rho\sqrt{1-\rho^2}\mathbb{E}SW + (1-\rho^2)\mathbb{E}W^2 = 1 + 2\rho^2.$$

Complex case
Use the real case for the real and imaginary parts taking into account that these r.v. have variance a half. $\square$

Computation of the covariances of the derivatives

Fix $z, w \in \mathbb{C}^{m+1}$. Let $\{v_2, \dots, v_m\}$ be an orthonormal set in $\mathbb{C}^{m+1}$ such that $\langle v_k, z\rangle = \langle v_k, w\rangle = 0$, $(k \geq 2)$. Define

$$v_z = \frac{w - \langle w, z\rangle z}{\sqrt{1 - |\langle z, w\rangle|^2}}, \qquad v_w = \frac{z - \langle z, w\rangle w}{\sqrt{1 - |\langle z, w\rangle|^2}}.$$

Then $B_z = \{v_z, v_2, \dots, v_m\}$ and $B_w = \{v_w, v_2, \dots, v_m\}$ are orthonormal basis of $z^\perp$ and $w^\perp$ respectively.

It is easy to see that

$$\langle z, v_w\rangle = \langle w, v_z\rangle = \sqrt{1 - |\langle z, w\rangle|^2}, \qquad \langle v_z, v_w\rangle = -\langle w, z\rangle.$$

Denote $\partial_k f(w)$ for $\frac{\partial f}{\partial v_k}(w)$, $k = z, w, 2, \dots, m$. and express all the derivatives on these basis.

$$f'(z) = \begin{pmatrix} \partial_z f_1(z) & \partial_2 f_1(z) & \dots & \partial_m f_1(z) \\ \partial_z f_2(z) & \partial_2 f_2(z) & \dots & \partial_m f_2(z) \\ \vdots & \vdots & \ddots & \vdots \\ \partial_z f_m(z) & \partial_2 f_m(z) & \dots & \partial_m f_(z) \end{pmatrix},$$

$$f'(w) = \begin{pmatrix} \partial_w f_1(w) & \partial_2 f_1(w) & \dots & \partial_m f_1(w) \\ \partial_w f_2(w) & \partial_2 f_2(w) & \dots & \partial_m f_2(w) \\ \vdots & \vdots & \ddots & \vdots \\ \partial_w f_m(w) & \partial_2 f_m(w) & \dots & \partial_m f_(w) \end{pmatrix}$$

Lemma 5.3.5. *Let $z, w \in \mathbb{C}^{m+1}$ be such that $\|z\| = \|w\| = 1$.*

	$k = z$	$k = w$	$k \geq 2$
$\mathbb{E}\,\partial_k f_\ell(w)\overline{f_\ell(w)}$	0		0
$\mathbb{E}\,\partial_k f_\ell(w)\overline{f_\ell(z)}$	$d_\ell \langle w, z\rangle^{d_\ell - 1}\sqrt{1 - \lvert\langle z, w\rangle\rvert^2}$		0
$\mathbb{E}\,\partial_k f_\ell(z)\overline{f_\ell(z)}$		0	0
$\mathbb{E}\,\partial_k f_\ell(z)\overline{f_\ell(w)}$		$d_\ell \langle z, w\rangle^{d_\ell - 1}\sqrt{1 - \lvert\langle z, w\rangle\rvert^2}$	0

Furthermore

$$\mathbb{E}\partial_s f_\ell(z)\overline{\partial_w f_\ell(w)} = d_\ell(d_\ell - 1)\langle z, w\rangle^{d_\ell - 2}(1 - |\langle z, w\rangle|^2) - d_\ell \langle z, w\rangle_\ell^d.$$

Proof. Since $\mathbb{E}\, f(z)\overline{f(z)} \equiv 1$ on the sphere and we take derivatives on the tangent space, these derivatives vanish. Besides $\langle z, v_k\rangle = \langle w, v_k\rangle = 0$ for $k \geq 2$.

Now

$$\begin{aligned}
\mathbb{E}f(z)\overline{\partial_w f(w)} &= \overline{\overline{\partial_w \langle w, z\rangle^d}} \\
&= d\langle w, z\rangle^{d-1}\frac{\partial}{\partial v_w}\langle w, z\rangle = d\langle z, w\rangle^{d-1}\langle z, v_w\rangle \\
&= d\langle z, w\rangle^{d-1}\sqrt{1 - |\langle z, w\rangle|^2}.
\end{aligned}$$

Taking derivative with respect to v_z we have

$$\begin{aligned}
\mathbb{E}\partial_z f(z)\overline{\partial_w f(w)} &= \frac{\partial}{\partial v_z}\left(d\langle z, w\rangle^{d-1}\langle s, v_t\rangle\right) \\
&= d(d-1)\langle z, w\rangle^{d-2}(1 - |\langle z, w\rangle|^2) - d\langle z, w\rangle^d
\end{aligned}$$

$\square$

Regression of $f'(w)$ over $f(z)$ and $f(w)$:

Choose $\alpha_{w\ell}, \beta_{w\ell}$ such that $\partial_w f(w) - \alpha f(w) - \beta f(z)$ be independent of $f(z), f(w)$. That is, α, β are the solution of the system:

$$\begin{cases} \alpha + \langle z, w\rangle^{d_\ell}\beta &= 0 \\ \langle w, z\rangle^{d_\ell}\alpha + \beta &= d_\ell \langle w, z\rangle^{d_\ell - 1}\langle z, v_w\rangle \end{cases}$$

Then

$$\alpha_{w\ell} = -\langle z, w\rangle^{d_\ell} \beta_{w\ell} \qquad \beta_{w\ell} = d_\ell \frac{\langle w, z\rangle^{d_\ell - 1} \langle z, v_w\rangle}{1 - |\langle z, w\rangle|^{2d_\ell}}.$$

The remaining $\alpha_{k\ell}, \beta_{k\ell}$ $(k \geq 2)$ vanish.

Regression of $f'(z)$ over $f(z)$ and $f(w)$:
The same arguments show that

$$\alpha_{1\ell} = -\langle w, z\rangle^{d_\ell} \beta_{1\ell} \qquad \beta_{1\ell} = d_\ell \frac{\langle z, w\rangle^{d_\ell - 1} \langle w, v_z\rangle}{1 - |\langle z, w\rangle|^{2d_\ell}}.$$

The remaining $\alpha_{k\ell}, \beta_{k\ell}$ $(k \geq 2)$ vanish.

Computation of τ and σ^2

$$\tau = \mathbb{E}\left(\partial_z f(z) - \alpha_s f(z) - \beta_s f(w)\right) \overline{\partial_w f(w)}$$

Then, by Lemma 5.3.5

$$\begin{aligned}
\tau &= d(d-1)\langle z, w\rangle^{d-2}(1 - |\langle z, w\rangle|^2) - d\langle z, w\rangle^{d-2}|\langle z, w\rangle|^2 + \\
&\qquad + d^2\langle z, w\rangle^{d-2}|\langle z, w\rangle|^{2d} \frac{1 - |\langle z, w\rangle|^2}{1 - |\langle z, w\rangle|^{2d}} \\
&= d\langle z, w\rangle^{d-2}\left[(d-1)(1 - |\langle z, w\rangle|^2) - |\langle z, w\rangle|^2 + d|\langle z, w\rangle|^{2d}\frac{1 - |\langle z, w\rangle|^2}{1 - |\langle z, w\rangle|^{2d}}\right] \\
&= d\langle z, w\rangle^{d-2}\left[-1 + d(1 - |\langle z, w\rangle|^2)\left(1 + \frac{|\langle z, w\rangle|^{2d}}{1 - |\langle z, w\rangle|^{2d}}\right)\right] \\
&= d\langle z, w\rangle^{d-2}\left[-1 + d\frac{1 - |\langle z, w\rangle|^2}{1 - |\langle z, w\rangle|^{2d}}\right].
\end{aligned}$$

That is, for each i we have

$$\tau_i = \langle z, w\rangle^{d_i - 2}\left[1 - \frac{d}{1 + |\langle z, w\rangle|^2 + \cdots + |\langle z, w\rangle|^{2d_i - 2}}\right]$$

Similarly,

$$\sigma^2 = \mathbb{E}\left(\partial_z f(z) - \alpha_s f(z) - \beta_s f(w)\right) \overline{\partial_z f(z)}$$

Again by Lemma 5.3.5 we obtain:

$$\sigma^2 = d\left[1 - d|\langle z, w\rangle|^{2d-2}\frac{(1-|\langle z, w\rangle|^2)}{1-|\langle z, w\rangle|^{2d}}\right],$$

and therefore for each i we get

$$\sigma_i^2 = 1 - \frac{d|\langle z, w\rangle|^{2d-2}}{1+|\langle z, w\rangle|^2+\cdots+|\langle z, w\rangle|^{2d_i-2}}.$$

Chapter 6

Minimizing the discrete logarithmic energy on the sphere: The role of random polynomials

In this chapter we prove that points in the sphere associated with roots of random polynomials via the stereographic projection, are surprisignly well-suited with respect to the minimal logarithmic energy on the sphere. That is, roots of random polynomials provide a fairly good approximation to Elliptic Fekete points. This chapter follows from a joint work with Carlos Beltrán and Michael Shub. (c.f. Armentano *et al.* [2011]).

6.1 Introduction and Main Result

This chapter deals with the problem of distributing points in the 2-dimensional sphere, in a way that the logarithmic energy is minimized. More precisely, let $x_1, \ldots, x_N \in \mathbb{R}^3$, and let

$$V(x_1, \ldots, x_N) = \ln \prod_{1 \le i < j \le N} \frac{1}{\|x_i - x_j\|} = - \sum_{1 \le i < j \le N} \ln \|x_i - x_j\| \qquad (6.1.1)$$

be the logarithmic energy of the N-tuple $x_1, \ldots, x_N$. Here, $\|\cdot\|$ is the Euclidean norm in $\mathbb{R}^3$. Let

$$V_N = \min_{x_1,\ldots,x_N \in \mathbb{S}^2} V(x_1, \ldots, x_N)$$

denote the minimum of this function when the x_k are allowed to move in the unit sphere $\mathbb{S}^2 = \{x \in \mathbb{R}^3 : \|x\| = 1\}$. We are interested in N-tuples minimizing the quantity (6.1.1). These optimal N-tuples are usually called Elliptic Fekete Points. This is a classical problem (see Whyte [1952] for its origins) that has attracted much attention during the last years. The reader may find modern background in Dragnev [2002], Kuijlaars & Saff [1998], Rakhmanov *et al.* [1994] and references therein. It is considered an example of highly non-trivial optimization problem. In the list of Smale's problems for the XXI Century Smale [2000], problem number 7 reads

Problem 1. *Can one find $x_1, \ldots, x_N \in \mathbb{S}^2$ such that*

$$V(x_1, \ldots, x_N) - V_N \leq c \ln N, \tag{6.1.2}$$

c a universal constant?

More precisely, Smale demands a real number algorithm in the sense of Blum *et al.* [1998] that with input N returns a N-tuple $x_1, \ldots, x_N$ satisfying equation (6.1.2), and such that the running time is polynomial on N.

One of the main difficulties when dealing with Problem 1 is that the value of V_N is not completely known. To our knowledge, the most precise result is the following, proved in [Rakhmanov *et al.*, 1994, Th. 3.1 and Th. 3.2].

Theorem 19. *Defining C_N by*

$$V_N = -\frac{N^2}{4} \ln\left(\frac{4}{e}\right) - \frac{N \ln N}{4} + C_N N,$$

we have

$$-0.112768770\ldots \leq \liminf_{N\to\infty} C_N \leq \limsup_{N\to\infty} C_N \leq -0.0234973\ldots$$

Thus, the value of V_N is not even known up to logarithmic precision, as required by equation (6.1.2).

The lower bound of Theorem 19 is obtained by algebraic manipulation of the formula for $V(x_1, \ldots, x_N)$, and the upper bound is obtained by the explicit construction of N-tuples $x_1, \ldots, x_N$ at which V attains small values.

In this chapter we choose a completely different approach to this problem. First, assume that $y_1, \ldots, y_N$ are chosen randomly and independently on the sphere, with the uniform distribution. One can easily show that the expected value of the function $V(y_1, \ldots, y_N)$ in this case is,

$$\mathbb{E}(V(y_1, \ldots, y_N)) = -\frac{N^2}{4} \ln\left(\frac{4}{e}\right) + \frac{N}{4} \ln\left(\frac{4}{e}\right). \tag{6.1.3}$$

Thus, a random choice of points in the sphere with the uniform distribution already provides a reasonable approach to the minimal value V_N, accurate to the order of $O(N \ln N)$. It is a natural question whether other handy probability distributions, i.e. different from the uniform distribution in $(\mathbb{S}^2)^N$, may yield better expected values. We will give a partial answer to this question in the framework of random polynomials.

Part of the motivation of Problem 1 is the search for a polynomial all of whose roots are well conditioned, in the context of Shub & Smale [1993c]. On the other hand, roots of random polynomials are known to be well conditioned, for a sensible choice of the random distribution of the polynomial (see Shub & Smale [1993b]). We make this connection more precise in the historical note at the end of the Introduction. This idea motivates the following approach:

Let f be a degree N polynomial. Let $z_1, \ldots, z_N \in \mathbb{C}$ be its complex roots. Let $z_k = u_k + iv_k$ and let

$$\hat{z}_k = \frac{(u_k, v_k, 1)}{1 + u_k^2 + v_k^2} \in \{x \in \mathbb{R}^3 : \|x - (0, 0, 1/2)\| = 1/2\}, \quad 1 \le k \le N, \tag{6.1.4}$$

be the associated points in the Riemann Sphere, i.e. the sphere of diameter 1 centered at $(0, 0, 1/2)$. Note that the $\hat{z}_k$'s are the inverse image under the stereographic projection of the z_k's, seen as points in the 2-dimensional plane

$\{(u,v,1) : u,v \in \mathbb{R}\}$. Finally, let

$$x_k = 2\hat{z}_k - (0,0,1) \in \mathbb{S}^2, \quad 1 \le k \le N, \tag{6.1.5}$$

be the associated points in the unit sphere. Note that the $\hat{z}_k, x_k$ depend only on f, so we can consider the two following mappings

$$f \mapsto V(\hat{z}_1,\ldots,\hat{z}_N), \qquad f \mapsto V(x_1,\ldots,x_N).$$

These two mappings are well defined in the sense that they do not depend on the way we choose to order the roots of f. Our main claim is that the points $x_1,\ldots,x_N$ are well-distributed for the function of equation (6.1.1), if the polynomial f is chosen with a particular distribution. That is, we will prove the following theorem in Section 6.2.

Theorem 20 (Main)**.** *Let $f(X) = \sum_{k=0}^{N} a_k X^k \in \mathcal{P}_N$ be a random polynomial, such that the coefficients a_k are independent complex random variables, such that the real and imaginary parts of a_k are independent (real) Gaussian random variables centered at* 0 *with variance $\binom{N}{k}$. Then, with the notations above,*

$$\mathbb{E}\left(V(\hat{z}_1,\ldots,\hat{z}_N)\right) = \frac{N^2}{4} - \frac{N\ln N}{4} - \frac{N}{4}.$$

$$\mathbb{E}\left(V(x_1,\ldots,x_N)\right) = -\frac{N^2}{4}\ln\left(\frac{4}{e}\right) - \frac{N\ln N}{4} + \frac{N}{4}\ln\frac{4}{e}.$$

By comparison of theorems 19 and 20 and equation (6.1.3), we see that the value of $V(x_1,\ldots,x_N)$ is surpringsingly small at points coming from the solution set of random polynomials! In figure 6.1 below we have plotted (using Matlab) the roots $z_1,\ldots,z_{70}$ and associated points $x_1,\ldots,x_{70}$ of a polynomial of degree 70 chosen randomly.

Equivalently, one can take random homogeneous polynomials (as in the historical note at the end of this introduction) and consider its complex projective solutions, under the identification of $\mathbb{P}(\mathbb{C}^2)$ with the Riemann sphere.

There exist different approaches to the problem of actually producing N-tuples satisfying inequality (6.1.2) above (see Bendito *et al.* [2009], Rakhmanov *et al.* [1994], Zhou [1995] and references therein), although none of them has been proved to solve Problem 1 yet. In Bendito *et al.* [2009] numerical experiments were done, designed to find local minima of the function V and involving massive computational effort. The method used there is a descent method which follows a gradient-like vector field. For the initial guess, N points are chosen at random in the unit sphere, with the uniform distribution.

Our Theorem 20 above suggests that better-suited initial guesses are those coming from the solution set of random polynomials. More especifically, consider the following numerical procedure:

1. Guess $a_k \in \mathbb{C}$, $k = 0 \ldots N$, complex random variables as in Theorem 20.

2. Construct the polynomial $f(X) = \sum_{k=0}^{N} a_k X^k$ and find its N complex solutions $z_1, \ldots, z_N \in \mathbb{C}$.

3. Construct the associated points in the unit sphere $x_1, \ldots, x_N$ following equations (6.1.4,6.1.5).

In view of Theorem 20, it seems reasonable for a flow-based search optimization procedure that attempts to compute optimal $x_1, \ldots, x_N$, to start by executing the procedure described above and then following the desired flow. Moreover, this procedure might solve Smale's problem on its own, as necessarily many random choices of the a_k's will produce values of V below the average and very close to V_N, possibly close enough to satisfy equation (6.1.2).

As it is well-known, item (2) of this procedure can only be done approximately. We may perform this task using some homotopy algorithm as the ones suggested in Beltrán & Pardo [2011], Shub [2009], Shub & Smale [1993a] which guarantee average polynomial running time, and produce arbitrarily close approximations to the z_k. In practice, it may be preferable to construct the companion matrix of f and to compute its eigenvalues with some standard Linear Algebra method.

The choice of the probability distribution for the coefficients of $f(X)$ in Theorem 20 is not casual. That probability distribution corresponds to the classical unitarily invariant Hermitian structure in the space of homogeneous polynomials, recalled at the beginning of Section 6.2 below (or see *Chapter 3*). This Hermitian structure is called by some authors Bombieri-Weyl structure, or Kostlan structure, and it is a classical construction with many interesting properties. The reader may see Blum *et al.* [1998] for background.

6.1.1 Historical Note

According to Smale [2000], part of the original motivation for Problem 1 was the search for well conditioned homogeneous polynomials as in Shub & Smale [1993c]. Given $g = g(X,Y)$ a degree N homogeneous polynomial with unknowns X, Y and complex coefficients, the condition number of g at a projective root $\zeta = (x,y) \in \mathbb{P}(\mathbb{C}^2)$ is defined by

$$\mu(g,\zeta) = N^{1/2} \frac{\|g\| \|\zeta\|^{N-1}}{|Dg(\zeta) \mid_{\zeta^\perp} |},$$

where $\|g\|$ is the Bombieri-Weyl norm of g and $Dg(\zeta) \mid_{\zeta^\perp}$ is the differential mapping of g at ζ, restricted to the complex orthogonal complement of ζ.

Let $f(X) = \sum_{k=0}^{N} a_k X^k$ be a degree N polynomial with one unknown X, and consider the homogeneous counterpart of f, $g(X,Y) = \sum_{k=0}^{N} a_k X^k Y^{N-k}$. The condition number $\mu(f,z)$ of f at a zero $z \in \mathbb{C}$ is then defined as $\mu(f,z) = \mu(g,(z,1))$.

Shub & Smale [1993b] proved that well-conditioned polynomials are highly probable. In Shub & Smale [1993c] the problem was raised as to how to write a deterministic algorithm which produces a polynomial g all of whose roots are well-conditioned. It was also realised that a polynomial whose projective roots (seen as points in the Riemann sphere) have logarithmic energy close to the minimum as in Smale's problem after scaling to $\mathbb{S}^2$, are well conditioned.

From the point of view of Shub & Smale [1993c], the ability to choose points at random already solves the problem. Here, instead of trying to use the logarithmic energy function $V(\cdot)$ to produce well-conditioned polynomials, we use the fact that random polynomials are well-conditioned, to try to produce low-energy N-tuples.

The relation between the condition number and the logarithmic energy is

$$V(\hat{z}_1, \ldots, \hat{z}_N) = \frac{1}{2}\sum_{i=1}^{N} \ln \mu(f, z_i) + \frac{N}{2}\sum_{i=1}^{N} \ln \sqrt{1+|z_i|^2} - \frac{N}{2}\ln \|f\| - \frac{N}{4}\ln N,$$

where the roots in $\mathbb{P}(\mathbb{C}^2)$ are $(z_i, 1)$, therefore f is monic.

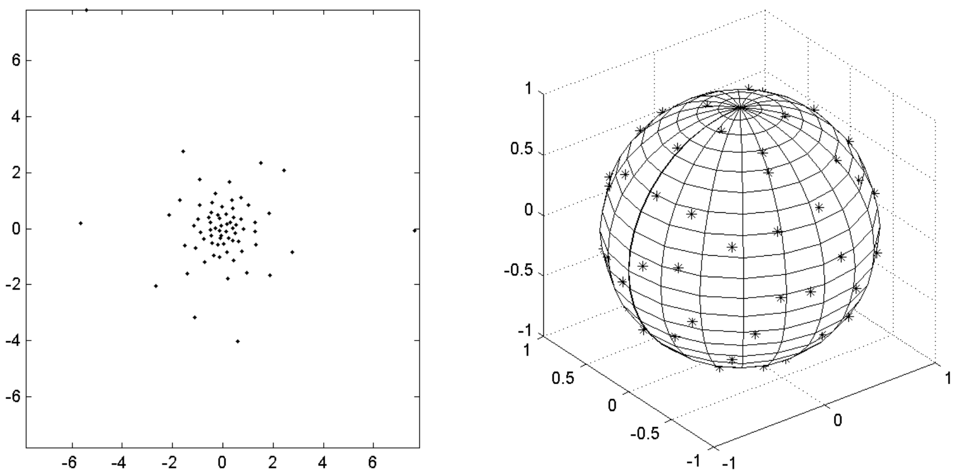


Figure 6.1: The points z_k and x_k for a degree 70 polynomial f chosen at random (using Matlab). The reader may see that the points in the sphere are pretty well distributed.

6.2 Technical tools and proof of Theorem 20

As in the introduction, $f = f(X)$ denotes a polynomial of degree N with complex coefficients, $z_1, \ldots, z_N \in \mathbb{C}$ are the complex roots of f, and $\hat{z}_1, \ldots, \hat{z}_N$ and $x_1, \ldots, x_N$ are the associated points in the Riemann Sphere and $\mathbb{S}^2$ respectively defined by equations (6.1.4,6.1.5). Let $\mathcal{P}_N$ be the vector space of degree N polynomials with complex coefficients. As in Beltrán & Pardo [2009a], Blum *et al.* [1998], we consider $\mathcal{P}_N$ endowed with the Bombieri-Weyl inner product, given by

$$\langle \sum_{k=0}^{N} a_k X^k, \sum_{k=0}^{N} b_k X^k \rangle = \sum_{k=0}^{N} \binom{N}{k}^{-1} a_k \overline{b_k}.$$

We denote the associated norm in $\mathcal{P}_N$ simply by $\|\cdot\|$. Let $f(X) = \sum_{k=0}^{N} a_k X^k$ be a random polynomial, where the a_k's are complex random variables as in Theorem 20. Then, note that the expected value of some measurable function $\phi : \mathcal{P}_N \to \mathbb{R}$ satisfies

$$\mathbb{E}(\phi(f)) = \frac{1}{(2\pi)^{N+1}} \int_{f \in \mathcal{P}_N} \phi(f) e^{-\|f\|^2/2} \, d\mathcal{P}_N. \tag{6.2.1}$$

Let $W = \{(f, z) \in \mathcal{P}_N \times \mathbb{C} : f(z) = 0\}$ be the so-called solution variety, which is a complex smooth submanifold of $\subseteq \mathcal{P}_N \times \mathbb{C}$ of dimension $N + 1$. For $z \in \mathbb{C}$, let $W_z = \{f \in \mathcal{P}_N : f(z) = 0\}$ be the set of polynomials which have z as a root. We consider W_z endowed with the inner product inherited from $\mathcal{P}_N$.

Proposition 6.2.1.

$$V(\hat{z}_1, \ldots, \hat{z}_N) = (N-1) \sum_{i=1}^{N} \ln \sqrt{1 + |z_i|^2} - \frac{1}{2} \sum_{i=1}^{N} \ln |f'(z_i)| + \frac{N}{2} \ln |a_N|,$$

Proof. A simple algebraic manipulation yields

$$V(\hat{z}_1, \ldots, \hat{z}_N) = -\sum_{1 \le i < j \le N} \ln \|\hat{z}_i - \hat{z}_j\| = -\sum_{1 \le i < j \le N} \ln \frac{|z_i - z_j|}{\sqrt{1 + |z_i|^2}\sqrt{1 + |z_j|^2}} =$$

$$(N-1) \sum_{i=1}^{N} \ln \sqrt{1 + |z_i|^2} - \sum_{1 \le i < j \le N} \ln |z_i - z_j|.$$

Note that

$$f(X) = a_N \prod_{i=1}^{N} (X - z_i).$$

Thus,

$$f'(z_i) = a_N \prod_{i \neq j} (z_i - z_j),$$

and

$$|a_N|^N \prod_{i=1}^{N} \frac{1}{|f'(z_i)|} = \prod_{i=1}^{N} \prod_{j \neq i} \frac{1}{|z_i - z_j|} = \prod_{1 \le i < j \le N} \frac{1}{|z_i - z_j|^2}.$$

Thus,

$$-\sum_{1 \le i < j \le N} \ln |z_i - z_j| = \frac{1}{2} \left(-\sum_{i=1}^{N} \ln |f'(z_i)| + N \ln |a_N| \right),$$

and the proposition follows. □

The rest of the proof of Theorem 20 will consist on the computation of the expected values of the quantities in Proposition 6.2.1. The following lemma will be useful

Lemma 6.2.1. *For any $t \in \mathbb{R}$,*

$$\sum_{k=0}^{N} \binom{N}{k} t^{2k} = (1+t^2)^N,$$

$$\sum_{k=1}^{N} \binom{N}{k} k t^{2k-1} = Nt(1+t^2)^{N-1},$$

$$\sum_{k=1}^{N} \binom{N}{k} k^2 t^{2k-2} = N(1+t^2)^{N-2}(1+Nt^2).$$

Proof. The first equality is the classical binomial expansion. Differentiate it to get

$$2\sum_{k=1}^{N} \binom{N}{k} k t^{2k-1} = 2Nt(1+t^2)^{N-1},$$

and the second equality follows. Differentiate again to get

$$\sum_{k=1}^{N} \binom{N}{k} (2k^2-k) t^{2k-2} = N(1+t^2)^{N-1} + 2N(N-1)t^2(1+t^2)^{N-2}.$$

Hence,

$$2\sum_{k=1}^{N} \binom{N}{k} k^2 t^{2k-2} = \frac{1}{t}\sum_{k=1}^{N} \binom{N}{k} k t^{2k-1} + N(1+t^2)^{N-1} + 2N(N-1)t^2(1+t^2)^{N-2} =$$

$$N(1+t^2)^{N-1} + N(1+t^2)^{N-1} + 2N(N-1)t^2(1+t^2)^{N-2} = 2N(1+t^2)^{N-2}(1+Nt^2).$$

The last equality of the lemma follows. □

Proposition 6.2.2. *Let $\phi : W \to \mathbb{R}$ be a measurable function. Then,*

$$\int_{f\in\mathcal{P}_N} \sum_{z:f(z)=0} \phi(f,z)\, d\mathcal{P}_N = \int_{z\in\mathbb{C}} \frac{1}{(1+|z|^2)^N} \int_{f\in W_z} |f'(z)|^2 \phi(f,z)\, dW_z\, d\mathbb{C} \tag{6.2.2}$$

Proof. As in [Blum *et al.*, 1998, Th. 5, p. 243], we apply the smooth coarea formula to the double fibration

$$\begin{array}{ccc} & W & \\ \swarrow & & \searrow \\ \mathcal{P}_N & & \mathbb{C} \end{array}$$

to get the formula

$$\int_{f\in\mathcal{P}_N} \sum_{z:f(z)=0} \phi(f,z)\, d\mathcal{P}_N = \int_{z\in\mathbb{C}} \int_{f\in W_z} (DG_z(f)DG_z(f)^*)^{-1} \phi(f,z)\, dW_z\, d\mathbb{C},$$

where $G_z : U_f \to U_z$ is the implicit function defined in a neighborhood of f satisfies $g(G_z(g)) = 0$, and $DG_z(f)$ is the Jacobian matrix of G_z at f, writen in some orthonormal basis. By implicit differentiation, $DG_z(f)\dot{f} = -f'(z)^{-1}\dot{f}(z)$. Thus, in the orthonormal basis given by the monomials $\binom{N}{k}^{1/2} X^k$, $k = 0\dots N$, the jacobian matrix is

$$DG_z(f) = -\frac{1}{f'(z)} \left(\binom{N}{0}^{1/2} z^0, \dots, \binom{N}{N}^{1/2} z^N \right).$$

We conclude that $DG_z(f)DG_z(f)^* = |f'(z)|^{-2} \sum_{k=0}^N \binom{N}{k} |z|^{2k} = |f'(z)|^{-2}(1 + |z|^2)^N$. The proposition follows. □

Proposition 6.2.3. *Let $z \in \mathbb{C}$ and let $\phi : \mathbb{R} \to \mathbb{R}$ be a measurable function. Then,*

$$\int_{f\in W_z} \phi(|f'(z)|^2) e^{-\|f\|^2/2}\, dW_z = (2\pi)^N \int_0^\infty t\phi\left(t^2 N(1+|z|^2)^{N-2}\right) e^{-t^2/2}\, dt.$$

Proof. Consider the mapping $\varphi : W_z \to \mathbb{C}$, $f(X) = \sum_{k=0}^N a_k X^k \mapsto w = f'(z) =$

$\sum_{k=0}^N ka_k z^{k-1}$. Denote by $NJ\varphi(f)$ the Normal Jacobian of φ at f, that is

$$NJ\varphi(f) = \max_{\dot{f}\in W_z, \|\dot{f}\|=1} \|D\varphi(f)\dot{f}\|^2$$

(see [Blum *et al.*, 1998, pag. 241] for references and background). Let $g_1, g_2 \in \mathcal{P}_N$ be the following polynomials,

$$g_1(X) = \sum_{k=0}^N \binom{N}{k} \overline{z}^k X^k, \quad g_2(X) = \sum_{k=1}^N k\binom{N}{k} \overline{z}^{k-1} X^k$$

Note that for any $f \in \mathcal{P}_N$ and $z \in \mathbb{C}$, we have

$$f(z) = \langle f, g_1\rangle, \quad f'(z) = \langle f, g_2\rangle.$$

Thus,

$$W_z = \{f \in \mathcal{P}_N : f(z) = 0\} = \{f \in \mathcal{P}_N : \langle f, g_1\rangle = 0\},$$
$$D\varphi(f)\dot{f} = \dot{f}'(z) = \langle \dot{f}, g_2\rangle.$$

Thus, if π is the orthogonal projection onto W_z, we have

$$NJ\varphi(f) = \max_{\dot{f}\in W_z, \|\dot{f}\|=1} |\langle \dot{f}, g_2\rangle|^2 = \|\pi(g_2)\|^2 = \|g_2\|^2 - \frac{|\langle g_1, g_2\rangle|^2}{\|g_1\|^2} =$$

$$\sum_{k=1}^N \binom{N}{k} k^2 |z|^{2k-2} - \frac{\left(\sum_{k=1}^N \binom{N}{k} k|z|^{2k-1}\right)^2}{\sum_{k=0}^N \binom{N}{k} |z|^{2k}}.$$

From Lemma 6.2.1, we conclude

$$NJ\varphi(f) = N(1+|z|^2)^{N-2}(1+N|z|^2) - \frac{N^2|z|^2(1+|z|^2)^{2N-2}}{(1+|z|^2)^N} =$$

$$N(1+|z|^2)^{N-2}(1+N|z|^2) - N^2|z|^2(1+|z|^2)^{N-2} = N(1+|z|^2)^{N-2}$$

The coarea formula [Blum *et al.*, 1998, p. 241] then yields

$$\int_{f\in W_z} \phi(|f'(z)|^2) e^{-\|f\|^2/2}\, dW_z = \tag{6.2.3}$$

$$\frac{1}{N(1+|z|^2)^{N-2}}\int_{w\in\mathbb{C}}\phi(|w|^2)\int_{\{f\in W_z:f'(z)=w\}}e^{-\|f\|^2/2}\,df\,d\mathbb{C}.$$

The set $\{f \in W_z : f'(z) = w\}$ is an affine subspace of $\mathcal{P}_N$ of dimension $N-1$, defined by the equations $\langle f, g_1\rangle = 0,\ \langle f, g_2\rangle = w$, which are linear independent equations on the coefficients of f. One can compute the norm of the minimal norm element of this affine subspace using standard tools from Linear Algebra. This minimal norm turns to be equal to $|w|\nu$ where

$$\nu = \frac{1}{\sqrt{\|g_2\|^2 - \frac{|\langle g_1,g_2\rangle|^2}{\|g_1\|^2}}} = \frac{1}{\sqrt{NJ\varphi(f)}} = \frac{1}{\sqrt{N}(1+|z|^2)^{\frac{N-2}{2}}}.$$

Thus,

$$\int_{\{f\in W_z:f'(z)=w\}}e^{-\|f\|^2/2}\,df = (2\pi)^{N-1}\exp\left(-\nu^2|w|^2/2\right),$$

and

$$\int_{w\in\mathbb{C}}\phi(|w|^2)\int_{f\in W_z:f'(z)=w}e^{-\|f\|^2/2}\,df\,d\mathbb{C} = (2\pi)^N\int_0^\infty \rho\phi(\rho^2)e^{-\nu^2\rho^2/2}\,d\rho =$$

$$\frac{(2\pi)^N}{\nu^2}\int_0^\infty t\phi\left(\frac{t^2}{\nu^2}\right)e^{-t^2/2}\,dt = (2\pi)^N N(1+|z|^2)^{N-2}\int_0^\infty t\phi\left(\frac{t^2}{\nu^2}\right)e^{-t^2/2}\,dt.$$

From this and equation (6.2.3) we conclude,

$$\int_{f\in W_z}\phi(|f'(z)|^2)e^{-\|f\|^2/2}\,dW_z = (2\pi)^N\int_0^\infty t\phi\left(\frac{t^2}{\nu^2}\right)e^{-t^2/2}\,d\rho,$$

as wanted. □

Proposition 6.2.4. *Let $f(X) = \sum_{k=0}^N a_k X^k$ where the a_k are as in Theorem 20. Then,*

$$\mathbb{E}\left(\sum_{i=1}^N \ln\sqrt{1+|z_i|^2}\right) = \frac{N}{2}. \tag{6.2.4}$$

$$\mathbb{E}\left(\ln|a_N|\right) = \frac{\ln(2)-\gamma}{2}. \tag{6.2.5}$$

$$\mathbb{E}\left(\sum_{i=1}^N \ln|f'(z_i)|\right) = \frac{(\ln(2)-1-\gamma+\ln(N)+N)N}{2}. \tag{6.2.6}$$

Here, $\gamma \sim 0.5772156649$ is Euler's constant.

Proof. From equalities (6.2.1,6.2.2),

$$\mathbb{E}\left(\sum_{i=1}^{N} \ln\sqrt{1+|z_i|^2}\right) = \frac{1}{(2\pi)^{N+1}} \int_{f\in\mathcal{P}_N} \sum_{i=1}^{N} \ln\sqrt{1+|z_i|^2} e^{-\|f\|^2/2}\, d\mathcal{P}_N =$$

$$\frac{1}{(2\pi)^{N+1}} \int_{z\in\mathbb{C}} \frac{\ln\sqrt{1+|z|^2}}{(1+|z|^2)^N} \int_{f\in W_z} |f'(z)|^2 e^{-\|f\|^2/2}\, dW_z\, d\mathbb{C}.$$

From Proposition 6.2.3,

$$\int_{f\in W_z} |f'(z)|^2 e^{-\|f\|^2/2}\, dW_z = (2\pi)^N \int_0^\infty t^3 N(1+|z|^2)^{N-2} e^{-t^2/2}\, dt =$$

$$(2\pi)^N 2N(1+|z|^2)^{N-2}.$$

Thus,

$$\mathbb{E}\left(\sum_{i=1}^{N} \ln\sqrt{1+|z_i|^2}\right) = \frac{N}{\pi} \int_{z\in\mathbb{C}} \frac{\ln\sqrt{1+|z|^2}}{(1+|z|^2)^2}\, d\mathbb{C} =$$

$$= 2N \int_0^\infty \frac{\rho\ln\sqrt{1+\rho^2}}{(1+\rho^2)^2}\, d\rho = \frac{N}{2},$$

and equation (6.2.4) follows. Equation (6.2.5) is trivial, as

$$\mathbb{E}\left(\ln|a_N|\right) = \frac{1}{2\pi} \int_{a\in\mathbb{C}} \ln|a| e^{-|a|^2/2}\, d\mathbb{C} = \int_0^\infty \rho\ln(\rho) e^{-\rho^2/2}\, d\rho = \frac{\ln(2)-\gamma}{2}.$$

Now let us prove equation (6.2.6). Note that from the equalities (6.2.1,6.2.2),

$$\mathbb{E}\left(\sum_{i=1}^{N} \ln|f'(z_i)|\right) = \frac{1}{(2\pi)^{N+1}} \int_{f\in\mathcal{P}_N} e^{-\|f\|^2/2} \sum_{z\in\mathbb{C}:f(z)=0} \ln|f'(z)|\, d\mathcal{P}_N =$$

$$\frac{1}{(2\pi)^{N+1}} \int_{z\in\mathbb{C}} \frac{1}{(1+|z|^2)^N} \int_{f\in W_z} e^{-\|f\|^2/2} |f'(z)|^2 \ln|f'(z)|\, dW_z\, d\mathbb{C} =$$

From Proposition 6.2.3, we know that

$$\int_{f\in W_z} |f'(z)|^2 \ln|f'(z)| e^{-\|f\|^2/2}\, dW_z =$$

$$(2\pi)^N \int_0^\infty t\left(t^2N(1+|z|^2)^{N-2}\right)\ln\sqrt{t^2N(1+|z|^2)^{N-2}}e^{-t^2/2}\,dt =$$

$$(2\pi)^N N(1+|z|^2)^{N-2}\int_0^\infty t^3\left(\ln t+\ln\sqrt{N(1+|z|^2)^{N-2}}\right)e^{-t^2/2}\,dt =$$

$$(2\pi)^N N(1+|z|^2)^{N-2}\left(1-\gamma+\ln 2+2\ln\sqrt{N(1+|z|^2)^{N-2}}\right).$$

Thus,

$$\mathbb{E}\left(\sum_{i=1}^N \ln|f'(z_i)|\right) = \frac{N}{2\pi}\int_{z\in\mathbb{C}}\frac{1-\gamma+\ln 2+\ln(N(1+|z|^2)^{N-2})}{(1+|z|^2)^2}\,d\mathbb{C} =$$

$$N\left(1-\gamma+\ln 2+\ln N\right)\int_0^\infty \frac{\rho}{(1+\rho^2)^2}\,d\rho + N(N-2)\int_0^\infty\frac{\rho\ln(1+\rho^2)}{(1+\rho^2)^2}\,d\rho =$$

$$\frac{N}{2}\left(1-\gamma+\ln 2+\ln N\right)+N\frac{N-2}{2},$$

and equation (6.2.6) follows.

□

6.2.1 Proof of Theorem 20

From Proposition 6.2.1,

$$\mathbb{E}\left(V(\hat{z}_1,\ldots,\hat{z}_N)\right) = (N-1)\mathbb{E}\left(\sum_{i=1}^N \ln\sqrt{1+|z_i|^2}\right)-\frac{1}{2}\mathbb{E}\left(\sum_{i=1}^N\ln|f'(z_i)|\right)+\frac{N}{2}\mathbb{E}\left(\ln|a_N|\right),$$

which from Proposition 6.2.4 is equal to

$$\frac{N(N-1)}{2}-\frac{(\ln(2)-1-\gamma+\ln(N)+N)N}{4}+\frac{N(\ln(2)-\gamma)}{4},$$

and the first assertion of Theorem 20 follows. The second equality of Theorem 20 is then trivial, as the affine transformation in $\mathbb{R}^3$ that takes the $\hat{z}_k$'s into the x_k's is a traslation followed by a homothety of dilation factor 2. Hence,

$$\|x_i-x_j\| = 2\|\hat{z}_i-\hat{z}_j\|,\quad 1\le i<j\le N,$$

and for any choice of $x_1, \ldots, x_N$ we have

$$V(x_1, \ldots, x_N) = V(\hat{z}_1, \ldots, \hat{z}_N) - \frac{N(N-1)}{2} \ln 2.$$

Appendices

Appendix B

Probability Theory

B.1 Gaussian distributions

Let $(\Omega, \mathcal{A}, P)$ be a probability space, that is, Ω is a set of "samples" provided by a σ-algebra $\mathcal{A}$, and $P : \mathcal{A} \to [0, 1]$ is a proabability measure (i.e. $P(\Omega) = 1$).

A measurable function $\eta : (\Omega, \mathcal{A}) \to (\mathbb{R}, \mathcal{B}_{\mathbb{R}})$ is called a *random variable* in $(\Omega, \mathcal{A})$. Here $\mathcal{B}_{\mathbb{R}}$ is the Borel σ-algebra in $\mathbb{R}$.

Given η a random variable in $(\Omega, \mathcal{A}, P)$, the *probability distribution* P_η associated to η is the push-forward mesure $\eta^* P = P \circ \eta^{-1}$, that is, the measure on $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$ given by $P_\eta(B) = P(\eta^{-1}(B))$ for all $B \in \mathcal{B}_{\mathbb{R}}$.

In this way, a random variable η in $(\Omega, \mathcal{A}, P)$ induces a probability space in $(\mathbb{R}, \mathcal{B}_{\mathbb{R}}, P_\eta)$.

We say that the random variable η is a *Gaussian random variable* centered at $\mu \in \mathbb{R}$ with variance $\sigma^2 > 0$, and we write $\eta \sim \mathcal{N}(\mu, \sigma^2)$, when the induced probability distribution P_η on $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$ is given by

$$P_\eta(B) = \frac{1}{\sqrt{2\pi}\sigma} \int_B e^{-(x-\mu)^2/(2\sigma^2)}\, dx, \qquad \text{for all} \quad B \in \mathcal{B}_{\mathbb{R}}.$$

A random vector is a n-tuple $\eta = (\eta_1, \ldots, \eta_n)$ whose components η_i are random variables on the same probability space $(\Omega, \mathcal{A}, P)$. *Mutatis mutandis*, the random vector η induces a probability measure on $(\mathbb{R}^n, \mathcal{B}_{\mathbb{R}^n}, P_\eta)$.

We say that the random vector $\eta = (\eta_1, \ldots, \eta_n)$ is a *Gaussian random vector* centered at $\mu \in \mathbb{R}^n$ with variance matrix $\mathrm{Var}(\eta) = \Sigma$ (positive definite), when the induced probability measure on $(\mathbb{R}^n, \mathcal{B}_{\mathbb{R}^n})$ is given by

$$P_\eta(B) = \frac{1}{\sqrt{2\pi}^n \sqrt{\det(\Sigma)}} \int_B e^{-\frac{1}{2}\langle \Sigma^{-1}(x-\mu), x-\mu\rangle} \, dx_1 \dots dx_n, \qquad \text{for all} \quad B \in \mathcal{B}_{\mathbb{R}^n}.$$

Here $\langle \cdot, \cdot \rangle$ is the scalar product in $\mathbb{R}^n$.

When $\mu = 0$ and $\Sigma = I_n$ we say that $\eta = (\eta_1, \dots, \eta_n)$ is a *standard Gaussian* in $\mathbb{R}^n$.

Remark:

- One can extend the definition of a Gaussian random vector when $\mathrm{Var}(\eta)$ is not positively definite. However in order to extend this definition one should introduce the Fourier transform. See for example Azaïs & Wschebor [2009].

- When $\Omega = \mathbb{R}^n$ the σ-algebra $\mathcal{A}$ is given by the Borel σ-algebra $\mathcal{B}_{\mathbb{R}^n}$.

B.2 Conditional Expectation

The conditional expectation is fairly known concept in probability and statistics. In the case that the random variables involved are Gaussian, the conditional expectation takes simple form:

Assume that ξ and η are random vectors on $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively and such that the distribution of $(\xi, \eta) \in \mathbb{R}^{m+n}$ is Gaussian. Assume also that $\mathrm{Var}(\eta)$ is positive definite. For simplicity, assume that ξ and η are centered.

Let $\varphi : \mathbb{R}^m \to \mathbb{R}$ be a bounded function, and suppose we want to compute

$$\mathbb{E}(\varphi(\xi) | \eta = y).$$

The idea is to choose a deterministic matrix C such that the random vectors $\zeta = \xi - C\eta$ and η become independent. That is, choose C such that

$$0 = \mathrm{Cov}(\xi - C\eta, \eta) := \mathbb{E}((\xi - C\eta)\eta^T) = \mathbb{E}(\xi\eta^T) - C\mathrm{Var}(\eta),$$

where a^T is the transpose of the column vector a.

Therefore,

$$\mathbb{E}(\varphi(\xi)|\eta = y) = \mathbb{E}(\varphi(\zeta + C\eta)|\eta = y) = \mathbb{E}(\varphi(\zeta + Cy)),$$

where ζ is a centered Gaussian variable with variance matrix

$$\mathrm{Var}(\xi) - \mathrm{Cov}(\xi, \eta)\mathrm{Var}(\eta)^{-1}\mathrm{Cov}(\xi, \eta)^T.$$

B.3 Stochastic Process and Random Fields

A *real valued stochastic process* indexed by the set I is collection of random variables $\mathcal{X} = \{X(t) : t \in I\}$ defined on a probability space $(\Omega, \mathcal{A}, P)$. In other words, a stochastic process is a function $X : \Omega \times I \to \mathbb{R}$, $X(\omega, t) = X(t)(\omega)$, such that is measurable in the first variable.

For a fixed $\omega \in \Omega$ the function $X(\omega, \cdot) : I \to \mathbb{R}$, given by $t \mapsto X(\omega, t)$, is a trajectory of the process. In this way, a stochastic process may be seen as a random "variable" taking values on a space of functions: $\omega \in \Omega \mapsto X(\omega, \cdot) \in \mathbb{R}^I$, where $\mathbb{R}^I$ is the set of functions from I to $\mathbb{R}$.

We say that a random process X is Gaussian is given any finite set of indexes $\{t_1, \ldots, t_k\}$, the random vector $(X(t_1), \ldots, X(t_k))$ is Gaussian.

When $\mathcal{X}$ is a collection of random vectors on $\mathbb{R}^k$, we say that $X : \Omega \times I \to \mathbb{R}^k$ is a *random field* or *stochastic fields*.

In the special case when $\Omega = \mathbb{R}^I$, the *canonical process* is given by $X(t)(\omega) = \omega(t)$.

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